Planar Spin Network Coherent States

II. Matrix Elements

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Abstract

This paper is the second of a series of three which construct coherent states for spin networks with planar symmetry. Paper 1 proposes a candidate set of coherent states. The present paper demonstrates explicitly that these states are approximate eigenvectors of the holonomy and momentum operators $\tilde{\mathrm{E}}^i_I$ (as expected for coherent states), up to small correction terms. Those correction terms are calculated. A complete subset of the overcomplete set of coherent states is constructed, and used to compute the approximate inverse of the volume operator. A theorem of Thiemann and Winkler is used to calculate the matrix elements of the [volume, holonomy] commutator. In the classical limit this commutator takes the derivative of volume with respect to angular momentum L.

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I Introduction

This paper is the second of a series of three which construct coherent states for spin networks with planar symmetry. The first paper of this series proposed a candidate set of coherent states for planar gravity waves [1]. Sections II and III of the present paper calculate the action of the densitized inverse triads $\tilde{\mathrm{E}}^i_I$ and the spin one-half holonomy on these states. In each case, the leading contribution to each matrix element is derived; and qualitative estimates are given for the size of the small correction (SC) terms. Quantitative measures for the size of SC terms are worked out in the appendices, especially appendix D. The third paper in the series will discuss the behavior of the Hamiltonian in the classical limit [2].

The coherent states form an overcomplete set, and as such are not well suited as a basis for perturbation theory. Section IV constructs a complete subset of the overcomplete set, and uses the subset to construct a series expansion for the inverse of the volume operator.

The spin network Hamiltonian depends only on the $\tilde{\mathrm{E}}$ and the spin 1/2 holonomy, and matrix elements of these operators are worked out in sections II and III. However, the Hamiltonian contains a commutator of volume with holonomy, suggested by Thiemann to regularize a dangerous factor of inverse triad determinant in the field theoretic Hamiltonian [3]. This commutator is sensitive to the small correction terms (SC terms) and is difficult to compute using only the results of sections II and III. However, a result due to Thiemann and Winkler does the trick [4]. Accordingly (sections IV discusses the volume operator and its inverse, then) section V derives a formula for the commutator, using the Thiemann Winkler result.

In appendix C I must restrict the magnitude p to be large, $e^p \gg 1$, say p greater than 5 or so, in order to obtain manageable approximations for the D(H) factors in the coherent state wavefunction. Whereas the matrix u determines peak values of the holonomy, therefore peak values of angle, the vector $\vec{p}$ determines peak value of the conjugate variable, angular momentum L. Specifically, from eq. (85), the magnitude p determines the peak value of the magnitude L of the angular momentum.

$$< \mathrm{L} + 1/2 >= \mathrm{p/t}.$$

Since we are focusing on large $< L >$, the restriction to large p is

not a problem.

The following is a summary of the most important formulas from paper 1. The proposed coherent states are sums of O(3) spherical harmonics.

$$
\begin{aligned}
&\mid u, \vec{p}\rangle = N \sum_{L,M} ((2L+1)/4\pi) \\
&\qquad\qquad \cdot \exp[-tL(L+1)/2] D^{(L)}(h)_{0M} D^{(L)}(g\dagger)_{M0}. \quad (1)
\end{aligned}
$$

The rotation matrix D(h) is a spherical harmonic, apart from normalization.

$$
\sqrt{4\pi/(2\mathrm{L}+1)}\,\mathrm{Y}_{\mathrm{LM}}(\theta/2, \phi-\pi/2) = \mathrm{D}^{(\mathrm{L})}(\mathrm{h})_{0\mathrm{M}}(-\phi+\pi/2, \theta/2, \phi-\pi/2). \quad (2)
$$

The matrix g, eq. (1), is in SL(2,C); every such matrix can be decomposed into a product of a Hermitean matrix H times a unitary matrix u.

$$
\begin{aligned}
g^{(L)} &= \exp[\vec{S} \cdot \vec{p}] u^{(L)} \\
&:= H^{(L)} u^{(L)} \qquad (3)
\end{aligned}
$$

All three matrices (h, H, and u) have their axis of rotation in the XY plane.

$$
\begin{aligned}
h^{(L)} &= \exp[i\,\hat{m} \cdot \vec{S}\,\theta/2]; \qquad (4) \\
\hat{m} &= (\cos\phi, \sin\phi, 0). \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
u^{(L)} &= u(-\beta+\pi/2, \alpha, \beta-\pi/2) \\
&:= \exp[i\,\hat{n} \cdot \vec{S}\,\alpha/2]; \\
\hat{n} &= (\cos\beta, \sin\beta, 0). \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
H^{(L)} &= \exp[\vec{S} \cdot \vec{p}]; \\
\vec{p} &= p\,[\cos(\beta+\mu), \sin(\beta+\mu), 0]. \qquad (7)
\end{aligned}
$$

From the qualitative discussion given in paper 1, the matrix u should determine the peak value of the holonomy h, while $\vec{p}$ should determine the peak angular momentum.

The Hamiltonian is expected to contain both densitized triads $\tilde{\mathrm{E}}$ and holonomies $h^{(1/2)}$ in the fundamental representation of SU(2).

However, it is desirable to replace the SU(2) representations in the Hamiltonian by O(3) harmonics, and construct the coherent states as a series of O(3) harmonics, because the action of the $\tilde{\mathrm{E}}$ on O(3) harmonics is especially simple; see for example section II of the present paper. In particular, matrix elements of the SU(2) holonomies in the Hamiltonian, $h_{mn}^{(1/2)}$, should be replaced by O(3) spherical harmonics, by using the formulas

$$\begin{aligned} \hat{V}(h)_{\pm} &= h_{\mp\pm}^{(1/2)}/\sqrt{2}; \\ \hat{V}(h)_0 &= (h_{++}^{(1/2)} + h_{--}^{(1/2)})/2; \\ \hat{V}(h)_M &= D^{(1)}(h)_{0M}. \end{aligned} \tag{8}$$

From the last line of eq. (8) the components of the unit vector $\hat{V}(h)$ form an L = 1 spherical harmonic of O(3). In section III on matrix elements of the spin 1/2 holonomy, I replace the holonomy by $\hat{V}(h)$, in effect replacing SU(2) by O(3). This replacement makes it easier to compute the action of the holonomy on the O(3) coherent states.

Eq. (1) can also represent the angular wavefunction of a particle moving in a central potential. In this interpretation, $\hat{V}(h)$ is the unit radius vector of the fictitious particle, and $\hat{V}(u)$ is the peak value of the radius vector.

II Action of the $\tilde{\mathrm{E}}$

This section computes the action of an $\tilde{\mathrm{E}}$ operator on the coherent state. The main results of this section are the following: the coherent states are approximate eigenvectors of the $\tilde{\mathrm{E}}$,

$$(\gamma\kappa/2)^{-1}\tilde{\mathrm{E}}_A^x \mid \mathrm{u}, \tilde{\mathrm{p}}\rangle = < \mathrm{L} > (\hat{\mathrm{n}}_\mathrm{A} \cos\mu - \hat{\mathrm{n}} \times \hat{\mathrm{V}})_\mathrm{A} \sin\mu) \mid \mathrm{u}, \tilde{\mathrm{p}}\rangle + \mathrm{SC}. \tag{9}$$

$< L >$ is the average, or peak value of the angular momentum. $\hat{n}$ is the axis of rotation for u, eq. (6), while μ is the angle between $\hat{n}$ and $\hat{p}$, eq. (7). $\hat{V}$ is the vector $\hat{V}(h)$, eq. (8), evaluated at the value h = u. (The next section on the holonomy will show that $\hat{V}$ is the peak value of $\hat{V}(h)$, hence u is the peak value of h.) The small correction terms SC are investigated in appendix D and shown to be down by factors of order $1/\sqrt{L}$.

I now derive eq. (9). By construction, the O(3) D(h) matrices in the coherent state transform simply under the action of an $\tilde{\mathrm{E}}$:

$$
\begin{aligned}
(\gamma\kappa/2)^{-1}\tilde{\mathrm{E}}^x_A \mid u,\vec{p}\rangle = N\sum_{L,M}((2L+1)/4\pi)\exp[-tL(L+1)/2] \\
\cdot D^{(L)}(h)_{0N}\langle L,N\mid S_A\mid L,M\rangle D^{(L)}(g\dagger)_{M0} \\
= N\sum_{L,M,R}((2L+1)/4\pi)\exp[-tL(L+1)/2] \\
\cdot D^{(L)}(h)_{0N}\langle L,N\mid S_A\mid L,M\rangle D^{(L)}(u\dagger)_{MR}D^{(L)}(H)_{R0} \\
= N\sum_{L}((2L+1)/4\pi)\exp[-tL(L+1)/2] \\
\cdot[\mathbf{D}^{(L)}(h)\mathbf{S}_A\mathbf{D}^{(L)}(u\dagger)\mathbf{D}^{(L)}(H)]_{00}
\end{aligned}
\tag{10}
$$

On the second line I use the g = Hu decomposition, eq. (3), plus $D(g\dagger) = D(u\dagger)D(H)$. On the last line I have used a matrix notation to hide some indices.

I now implement the general procedure outlined in section IV of paper 1. This procedure involves three steps. Produce a factor of $D^{(1)}(u)$, because the rows of this matrix provide a convenient basis of unit vectors; rewrite the summand until it looks as much as possible like the summand in the original coherent state, because we wish to prove that $\mid u,\vec{p}\rangle$ is an approximate eigenfunction of $\tilde{\mathrm{E}}$; simplify the summand by power series expanding and integrating term by term, because D(H) is Gaussian, and higher powers in the expansion will be suppressed.

First produce a factor of $D^{(1)}(u)$. The spin generator S in eq. (10) is essentially a Clebsch-Gordan coefficient, and from the rotation properties of these coefficients, appendix A, eq. (65),

$$
\mathbf{S}_{\mathrm{A}}\mathbf{D}^{(\mathrm{L})}(\mathrm{u}\dagger) = \mathbf{D}^{(\mathrm{L})}(\mathrm{u}\dagger)\mathbf{S}_{\mathrm{B}}\mathrm{D}^{(1)}(\mathrm{u})_{\mathrm{BA}}.
$$

Eq. (10) becomes

$$(\gamma\kappa/2)^{-1}\tilde{\mathrm{E}}^x_A \mid u,\vec{p}\rangle = N\sum_{L,M}((2L+1)/4\pi)\exp[-tL(L+1)/2]$$
$$\cdot[\mathbf{D}^{(L)}(hu\dagger)\mathbf{S}_B\mathbf{D}^{(L)}(H)]_{00}D^{(1)}(u)_{BA}$$
$$= N\sum_{L,M}((2L+1)/4\pi)\exp[-tL(L+1)/2]$$
$$\cdot D^{(L)}(hu\dagger)_{0M}D^{(L)}(H)_{M0}[\langle L,M \mid S_B \mid L,M-B\rangle D^{(L)}(H)_{M-B,0}/D^{(L)}(H)_{M0}]$$
$$\cdot D^{(1)}(u)_{BA}. \tag{11}$$

The final two lines exhibit the desired factor of $D^{(1)}(u)$, as well as factors of $D^{(L)}(hu\dagger)_{0M}D^{(L)}(H)_{M0}$ which make the sum in eq. (11) look as much as possible like the original coherent state.

I must now evaluate and expand the square bracket, second line from the end of eq. (11). The ratio of D(H) matrices follows from eq. (76) of appendix C, valid for $e^p \gg 1$.

$$\mathrm{D^{(L)}(H)_{M\mp1,0}} \cong \mathrm{D^{(L)}(H)_{M,0}}[1/(\hat{\mathrm{p}}_1 - \mathrm{i}\hat{\mathrm{p}}_2)]^{\pm1}\sqrt{(\mathrm{L}\pm\mathrm{M})/(\mathrm{L}\mp\mathrm{M}+1)}. \tag{12}$$

I have specialized to $p_3 = 0$. For this value of p_3 I can parameterize $\vec{p}$ as at eq. (7), so that $\hat{p}_1 - i\hat{p}_2 = \exp[-i(\beta+\mu)]$. Also, the matrix element of S_0 is just M; while S_B, B $= \pm1$, is given by

$$\langle \mathrm{L,M} \mid \mathrm{S}_{\pm1} \mid \mathrm{L,M}\mp1\rangle = \mp\sqrt{(\mathrm{L}\mp\mathrm{M}+1)(\mathrm{L}\pm\mathrm{M})/2} \tag{13}$$

Eq. (11) becomes

$$(\gamma\kappa/2)^{-1}\tilde{\mathrm{E}}^x_A \mid u,\vec{p}\rangle = N\sum_{L,M,B}((2L+1)/4\pi)\exp[-tL(L+1)/2]$$
$$\cdot D^{(L)}(hu\dagger)_{0M}C[M,B]D^{(L)}(H)_{M0}\exp[iB(\beta+\mu)]D^{(1)}(u)_{BA};$$
$$\begin{aligned} C[M,B=0] &= M; \\ C[M,B=+1] &= -(L+M)/\sqrt{2}; \\ C[M,B=-1] &= (L-M)/\sqrt{2}. \end{aligned} \tag{14}$$

I replace the $D^{(1)}(u)_{BA}$ by the geometrically more transparent quantities $\hat{n}$, $\hat{V}$, and $\hat{n}\times\hat{V}$. The row $D^{(1)}(u)_{0A}$ is just the vector $\hat{V}(h)$ introduced at eq. (8), with h replaced by u.

$$\mathrm{D}^{(1)}(\mathrm{u})_{0\mathrm{A}} = \hat{\mathrm{V}}(\mathrm{u})_{\mathrm{A}}.$$

The rows $D^{(1)}(u)_{\pm 1,A}$ may be replaced by linear combinations of the unit vectors $\hat{n}$ and $\hat{n} \times \hat{V}$, using eqs. (69) and (70) in appendix B. Eq. (14) becomes

$$\begin{aligned}(\gamma\kappa/2)^{-1}\tilde{\mathrm{E}}^x_A \mid u, \vec{p}\rangle = N \sum_{L,M} &((2L+1)/4\pi) \exp[-tL(L+1)/2] \\ &\cdot D^{(L)}(hu\dagger)_{0M} D^{(L)}(H)_{M0} \\ \cdot \{(L\cos\mu + iM\sin\mu)\hat{n}_A &+ (-L\sin\mu + iM\cos\mu)(\hat{n}\times\hat{V})_A \\ &+ M\hat{V}(u)_A\}. \quad (15)\end{aligned}$$

The third and final step of the general procedure invokes the Gaussian nature of the factor D(H). From appendix C

$$\begin{aligned}D^{(L)}(H)_{M0} \exp[-tL(L+1)/2] \cong \\ &\sqrt{t/\pi}\exp[-t(L+1/2-p/t)^2/2] \\ &\sqrt{1/[(L+1/2)\pi]}\exp\{-M^2/[2(L+1/2)]\}f(p,t).\end{aligned}$$

This suggests expanding the factors of L and M in the curly bracket around their means, $< L >=$ p/t -1/2 and $< M > = 0$. Equivalently, in eq. (15) set

$$\mathrm{L} =< \mathrm{L} > + \Delta\mathrm{L};\ \mathrm{M} = \Delta\mathrm{M},$$

and keep out to first order in ΔL and ΔM. The dominant terms in the expansion come from the terms in the square bracket involving $< L >$,

$$< \mathrm{L} > (\hat{\mathrm{n}}\cos\mu - \hat{\mathrm{n}} \times \hat{\mathrm{V}})_{\mathrm{A}} \sin\mu).$$

This expression is a constant and may be taken out of the sum, leaving behind just the original coherent state. We get the desired result eq. (9) for the leading term.

The remaining terms in the expansion are ΔX terms, X = L or M. These small correction (SC) terms are studied in appendix D, where it is shown they are down by factors of order $\sigma_X/ < L >$, σ_X the standard deviation for the Gaussian distribution of X.

III Matrix Elements of the Holonomy

This section computes the action of a holonomy $h^{(1/2)}$ on a coherent state. The main results are the following. When h acts on a coherent state, *all* of the states which result differ from the original coherent state. In particular, the leading terms are not identically equal to the original ket. To expand the action of the holonomy, it is necessary to introduce new kets

$$\begin{aligned} \mid L \pm 1\rangle &= N(L \pm 1) \sum_{L,M} ((2L+1)/4\pi) \exp[-tL(L+1)/2] \\ & \cdot D^{(L\pm 1)}(hu\dagger)_{0M} D^{(L)}(H)_{M0}. \end{aligned} \tag{16}$$

The first part of this section shows that these kets are Gaussian distributed in L, but with peak values of L shifted by one unit, from $< L >$ to $< L > \pm 1$. (Strictly speaking the ket notation should be $< L > \pm 1$ rather than L $\pm$ 1.) The second part of this section expresses the action of the holonomy in terms of these states and shows that

$$\mathrm{D}^{(1)}(\mathrm{h})_{0\mathrm{A}} \mid \mathrm{u}, \tilde{\mathrm{p}}\rangle = \mathrm{D}^{(1)}(\mathrm{u})_{0\mathrm{A}} (\mid \mathrm{L}+1\rangle + \mid \mathrm{L}-1\rangle)/2 + \mathrm{SC}. \tag{17}$$

From eq. (8), the O(3) spherical harmonics $D^{(1)}(h)_{0A}$ are linear combinations of $h^{1/2}$ matrix elements. Eq. (17) states that $D^{(1)}(u)_{0A}$ is the peak value of $D^{(1)}(h)_{0A}$; consequently u is the peak value of h. The linear combination $(\mid L+1\rangle + \mid L-1\rangle)/2$ is approximately the original ket $\mid u, \vec{p}\rangle$.

Section III of paper 1 describes a electron-in-a-central-force analogy which predicts that the expectation values of holonomy and $\tilde{\mathrm{E}}$ should be orthogonal.

$$0 = \sum_{\mathrm{A}} < \mathrm{D}^{(1)}(\mathrm{h})_{0\mathrm{A}} >< \tilde{\mathrm{E}}^x_A > . \tag{18}$$

From eq. (8), $D^{(1)}(u)_{0A}$ in eq. (17) is just $\hat{V}_A$. Therefore the constraint given in eq. (18) is obeyed, since $< \tilde{\mathrm{E}} >$, eq. (9), contains only vectors perpendicular to $\hat{V}$.

A the kets $\mid L \pm 1\rangle$

The states defined at eq. (16) have peak angular momentum shifted by one unit. Proof: in eq. (16), the D(H) factor is easier to approximate, while $D(hu\dagger)$ is harder. Therefore relabel $L{\pm}1 = \tilde{L}$, then drop tildes, in order to make the $D(hu\dagger)$ factor equal to the corresponding factor in the original coherent state $\mid u, \vec{p}\rangle$: under the relabeling, $D^{(L\pm1)}(hu\dagger){\rightarrow}D^{(\tilde{L})}(hu\dagger){\rightarrow}D^{(L)}(hu\dagger)$. The relabeling changes the exponential $\exp[-tL(L+1)/2]$ significantly; and it produces negligible changes in $(2L+1)/4\pi \cong (2\bar{L}+1)/4\pi + \mathrm{order}(1/L)$ The new D(H) factor is $D^{(L)}(H){\rightarrow}D^{(\tilde{L}\mp1)}(H){\rightarrow}D^{(L\mp1)}(H)$. This matrix differs from the corresponding matrix in $\mid u, \vec{p}\rangle$, but is easy to relate to that matrix. I group together the two factors in eq. (16) that change significantly, and use eq. (85).

$$\begin{aligned}
&\exp[-t(L\mp1)(L\mp1+1)/2]D^{(L\mp1)}(H)_{M0}\\
&\cong \exp\{-t[L\mp1+1/2]^2/2+t/8\}\exp[-iM\beta](\exp(p/2)/2)^{2L\mp2}\\
&\qquad\cdot\frac{(2L\mp2)!}{\sqrt{(L\mp1)!^2(L\mp1+M)!(L\mp1-M)!}}\\
&\qquad\cong \exp\{-t[L\mp1+1/2-p/t]^2/2+p^2/2t-p/2\}\\
&\qquad\exp\{-M^2/[2(L\mp1+1/2)]\}/\sqrt{\pi(L\mp1+1/2)}\\
&\qquad\cong \exp\{-t[L\mp1+1/2-p/t]^2/2+p^2/2t-p/2\}\\
&\qquad\qquad\exp\{-M^2/[2(L+1/2)]\}/\sqrt{\pi(L+1/2)}.
\end{aligned}\tag{19}$$

The factorials were replaced by a Gaussian using eq. (81). Now compare the Gaussian in eq. (19) to the Gaussian in the original coherent state, eq. (85). The two Gaussians in L are the same except for a shift in peak value.

$$\mathrm{new} < \mathrm{L}+1/2 >= \mathrm{p/t} \pm 1 = \mathrm{original} < \mathrm{L}+1/2 > \pm 1. \tag{20}$$

$\square$

B Action of the holonomy

Presumably the Hamiltonian will be a function of the matrix $h^{(1/2)}$; but for calculations it is more convenient to use the linear combinations $D^{(1)}(h)$, eq. (8), rather than the $h^{(1/2)}$. The coherent state is

an expansion in spherical harmonics $D^{(L)}(h)$, and using the matrix $D^{(1)}(h)$ allows one to invoke identities for simplifying the product of two D's.

$$
\begin{aligned}
D^{(1)}(h)_{0A} \mid u, \vec{p}\rangle = N \sum_{L,M} & ((2L+1)/4\pi) \exp[-tL(L+1)/2] \\
& \cdot \sum_{L\pm 1} D^{(L\pm 1)}(h)_{0M} \langle L\pm 1, 0 \mid L, 0; 1, 0\rangle \\
& \qquad \cdot \langle L\pm 1, M \mid L, M-A; 1, A\rangle \\
& \qquad\qquad \cdot D^{(L)}(u\dagger)_{M-A,N} D^{(L)}(H)_{N0} \quad (21)
\end{aligned}
$$

On the right I have rewritten $D(g\dagger) = D(u\dagger)D(H)$, from eq. (3); and I have used the formula for combining two D(h) matrices.

$$
\begin{aligned}
D^{(L)}(h)_{0,M-A} D^{(1)}(h)_{0A} = & \sum_{L\pm 1} D^{(L\pm 1)}(h)_{0M} \\
& \cdot \langle L\pm 1, 0 \mid L, 0; 1, 0\rangle \langle L\pm 1, M \mid L, M-A; 1, A\rangle .
\end{aligned}
$$

This formula is a special case of the general relation for coupling two D's, eq. (64). Note the value L is excluded from the sum on the right. $\langle L', 0 \mid L, 0; 1, 0\rangle$ vanishes for $\mathrm{L}' = \mathrm{L}$. Parity conservation also predicts no contribution from $\mathrm{L}' = \mathrm{L}$.

I now implement the same three step procedure as used in the previous section for Ẽ matrix elements. I introduce the basis vectors $D^{(1)}(u)$ by invoking eq. (64).

$$
\begin{aligned}
& \langle L\pm 1, M \mid L, M-A; 1, A\rangle D^{(L)}(u\dagger)_{M-A,N} \\
& \qquad = D^{(L\pm 1)}(u\dagger)_{M,N+B} \langle L\pm 1, N+B \mid L, N; 1, B\rangle D^{(1)}(u)_{BA} .
\end{aligned}
$$

I insert this into eq. (21), and relabel $N \to N - B$.

$$
\begin{aligned}
&D^{(1)}(h)_{0A} \mid u, \vec{p}\rangle \\
&\qquad = N \sum_{L,N} ((2L+1)/4\pi) \exp[-tL(L+1)/2] \\
&\qquad\qquad \cdot \sum_{L\pm 1} D^{(L\pm 1)}(hu\dagger)_{0N} D(H)_{N0} \\
&\qquad \cdot [\langle L \pm 1, 0 \mid L, 0; 1, 0\rangle \langle L \pm 1, N \mid L, N-B; 1, B\rangle \\
&\qquad\qquad \cdot D^{(L)}(H)_{N-B,0}/D(H)_{N,0}] D^{(1)}(u)_{BA}. \quad (22)
\end{aligned}
$$

I have multiplied and divided by $D(H)_{N0}$, so that the second and third lines resemble the original coherent state, except for the change $\mathrm{L} \rightarrow \mathrm{L} \pm 1$.

I must now evaluate and expand the square bracket on the last two lines. I use eqs. (71) and (76) with $p_3 = 0$ to evaluate the ratio

$$
\mathrm{D(H)_{N-B,0}/D(H)_{N0}}.
$$

I relabel $\mathrm{N} \rightarrow \mathrm{M}$, and use the results of appendix B to replace $D^{(1)}(u)$ by $\hat{n}$, $\hat{V}$, and $\hat{n} \times \hat{V}$.

$$
\begin{aligned}
&D^{(1)}(h)_{0,A} \mid u, \vec{p}\rangle \\
&\qquad = N \sum_{L,M} (2L+1)/(4\pi) \exp[-tL(L+1)/2] \\
&\qquad\qquad \cdot \sum_{L\pm 1} D^{(L\pm 1)}(hu\dagger)_{0M} D^{(L)}(H)_{M0} \\
&\qquad\qquad \cdot (c(L \pm 1, V)\hat{V}(u)_A + c(L \pm 1, n)\hat{n}_A \\
&\qquad\qquad\qquad + c(L \pm 1, \hat{n} \times \hat{V})(\hat{n} \times \hat{V})_A). \quad (23)
\end{aligned}
$$

The coefficients c are

$$
\begin{aligned}
c(L+1,V) &= \langle L+1,0 \mid L,0;1,0\rangle\langle L+1,M \mid L,M;1,0\rangle;\\
c(L+1,n) &= \langle L+1,0 \mid L,0;1,0\rangle\\
&\cdot\sum_{\pm}((\mp 1/\sqrt{2})\langle L+1,M \mid L,M\mp 1;1,\pm 1\rangle\\
&\cdot\exp(\pm i\mu)\sqrt{(L\pm M)/(L\mp M+1)});\\
c(L+1,\hat{n}\times\hat{V}) &= i\langle L+1,0 \mid L,0;1,0\rangle \qquad (24)\\
&\cdot\sum_{\pm}((-1/\sqrt{2})\langle L+1,M \mid L,M\mp 1;1,\pm 1\rangle\\
&\cdot\exp(\pm i\mu)\sqrt{(L\pm M)/(L\mp M+1)}). \qquad (25)
\end{aligned}
$$

For L+1 $\rightarrow$ L-1, replace $\langle L+1,M \mid$ and$\langle L+1,0 \mid$ by $\langle L-1,M \mid$ and$\langle L-1,0 \mid$. After some work with a table of 3J symbols, one finds

$$
\begin{aligned}
c(L+1,V) &= \sqrt{(L+1)^2-M^2}/(2L+1);\\
c(L-1,V) &= \sqrt{L^2-M^2}/(2L+1);\\
c(L+1,n) &= -[\sqrt{(L+1+M)/(L+1-M)}(L+M)/(2(2L+1))\\
&\quad -(M\rightarrow -M)]\cos\mu\\
&\quad -i[\sqrt{(L+1+M)/(L+1-M)}(L+M)/(2(2L+1))\\
&\quad +(M\rightarrow -M)]\sin\mu;\\
c(L-1,n) &= i[\sqrt{L^2-M^2}/(2L+1)]\sin\mu;\\
c(L+1,\hat{n}\times\hat{V}) &= -i[\sqrt{(L+1+M)/(L+1-M)}(L+M)/(2(2L+1))\\
&\quad +(M\rightarrow -M)]\cos\mu\\
&\quad +[\sqrt{(L+1+M)/(L+1-M)}(L+M)/(2(2L+1))\\
&\quad -(M\rightarrow -M)]\sin\mu;\\
c(L-1,\hat{n}\times\hat{V}) &= i[\sqrt{L^2-M^2}/(2L+1)]\cos\mu. \qquad (26)
\end{aligned}
$$

Continuing with the general procedure, I expand the L and M dependence of the c's. I replace

$$\text{L}\rightarrow <\text{L}> +\Delta\text{L};\ \text{M}\rightarrow\Delta\text{M}.$$

and keep terms in eq. (26) out to linear in ΔX, X = L or M.

$$
\begin{aligned}
c(L+1,V) &\cong <(L+1)/(2L+1)>; \\
c(L-1,V) &\cong <L/(2L+1)>; \\
c(L+1,n) &\cong -(M/<L>)\cos\mu - i\sin\mu <L/(2L+1)>; \\
c(L-1,n) &= i\sin\mu <L/(2L+1)>; \\
c(L+1,\hat{n}\times\hat{V}) &\cong -i<L/(2L+1)>\cos\mu + (M/<L>)\sin\mu; \\
c(L-1,\hat{n}\times\hat{V}) &\cong +i<L/(2L+1)>\cos\mu. \qquad (27)
\end{aligned}
$$

Factors of L inside brackets $<>$ should be interpreted as $<L>$; terms of order $\Delta X/<L>$ have been kept, but not terms of order $\Delta X/<L>^2$ or $(\Delta X/<L>)^2$. After inserting this expansion into eq. (23), that equation becomes

$$
\begin{aligned}
D^{(1)}(h)_{0A} \mid u,\vec{p}\rangle \cong\; & N\sum_{L,M}((2L+1)/4\pi)\exp[-tL(L+1)/2] \\
& \cdot\{(1/2)[D^{(L+1)}(hu\dagger)+D^{(L-1)}(hu\dagger)]_{0M}\hat{V}_A \\
& +D^{(L+1)}(hu\dagger)_{0M}(M/<L>) \\
& \cdot[-\cos\mu\hat{n}_A+\sin\mu(\hat{n}\times\hat{V})_A] \\
& -i(1/2)[D^{(L+1)}(hu\dagger)-D^{(L-1)}(hu\dagger)]_{0M} \\
& \cdot[\sin\mu\hat{n}_A+\cos\mu(\hat{n}\times\hat{V})_A] \\
& +\text{order}((M/L)^2,1/L)\}D^{(L)}(H)_{M0}. \qquad (28)
\end{aligned}
$$

I anticipate that

$$
\mathrm{M}/<\mathrm{L}>= \text{order}\,\sigma_{\mathrm{M}}/<\mathrm{L}>= \text{order}\,1/\sqrt{<\mathrm{L}>}.
$$

Therefore I am keeping order $1/\sqrt{<L>}$ but neglecting order $(M/L)^2$ = order 1/L. Therefore when going from eq. (27) to eq. (28) it is legitimate to replace $<L/(2L+1)>$ by 1/2, etc..

Now separate eq. (28) into a leading term plus small corrections. Eq. (28) involves the linear combinations $(1/2)[D^{(L+1)}(hu\dagger)\pm D^{(L-1)}(hu\dagger)]$, therefore linear combinations $(1/2)(\mid L+1\rangle\pm\mid L-1\rangle)$ of the kets investigated earlier in this section. The linear combination with the upper sign is a sum of two Gaussians in L. The two Gaussians have slightly different means, but large standard deviations $\sigma_L = 1/\sqrt{t}$, from eq. (19). Therefore the two Gaussians

strongly overlap with each other and with the Gaussian of the original coherent state $\mid u, \vec{p}\rangle$. Therefore, inside a sum over L,

$$\begin{aligned} (1/2)[D^{(L+1)}(hu\dagger) + D^{(L-1)}(hu\dagger)]D^{(L)}(H) &\cong D^{(L)}(hu\dagger)D^{(L)}(H); \\ (1/2)(\mid L+1\rangle + \mid L-1\rangle) &\cong \mid u, \vec{p}\rangle. \end{aligned} \tag{29}$$

The linear combination with the upper sign will yield the dominant term.

The linear combination with the lower sign, on the other hand, will be the *difference* of two very similar Gaussians, therefore will be very small. A detailed calculation in appendix D shows it to be a small correction.

Also, the $\hat{n}$ terms involve a factor of M/$< L >$, and the discussion of appendix D shows these terms are down by $1/\sqrt{L}$, as expected because $D^{(L)}(H)$ is Gaussian in M. Therefore only the $\hat{V}_A$ term is dominant. From eq. (8) $\hat{V}_A$ is also $D^{(1)}(u)_{0A}$; and we get precisely eq. (17).

In the leading, $\hat{V}$ term of eq. (28) I have not replaced the linear combination $(1/2)(D^{(L+1)}(hu\dagger) + D^{(L-1)}(hu\dagger))$ by $D^{(L)}(hu\dagger)$, even though $(1/2)(\mid L+1\rangle + \mid L-1\rangle)$ overlaps strongly with $\mid u, \vec{p}\rangle$. (Similarly, I have not replaced $(1/2)(\mid L+1\rangle + \mid L-1\rangle)$ by $\mid u, \vec{p}\rangle$ in eq. (17).) The linear combination $(1/2)(\mid L+1\rangle + \mid L-1\rangle)$ has the same $\tilde{\mathrm{E}}$ eigenvalues as $\mid u, \vec{p}\rangle$. However, $(1/2)(\mid L+1\rangle + \mid L-1\rangle)$ does not behave like $\mid u, \vec{p}\rangle$ under the action of a holonomy:

$$\begin{aligned} & D^{(1)}(h)_{0,A}(1/2)(\mid L+1\rangle + \mid L-1\rangle) \\ & \qquad \cong (1/4)(\mid L+2\rangle + 2 \mid L\rangle + \mid L-2\rangle), \end{aligned}$$

where $\mid L\rangle$ is the original state $\mid u, \vec{p}\rangle$. I am not yet sure whether these changes in peak L have any significance, and for now, I prefer to keep them explicit. Note if the Hamiltonian contains a product of n holonomies, then the original coherent state will be changed into a linear combination of coherent states with peak L shifted by as much as $\pm n$.

So far I have carried out all calculations assuming the axis of rotation for the matrix u and the vector $\vec{p}$ both lie in the XY plane. What happens if I replace (u, $\vec{p}$) by parameters $(\tilde{u}, \vec{\tilde{p}})$ not obeying this constraint? I state the following theorems without proof.

The new and old matrices H and $\tilde{\mathrm{H}}$ are connected by a similarity transformation.

$$\tilde{\mathrm{H}} = \mathrm{v}\dagger\mathrm{H}\mathrm{v}.$$

The matrix v is uniquely determined by $\vec{p}$ and $\tilde{\vec{p}}$, up to an overall sign:

$$\mathrm{v} = \exp[\mathrm{i}\sigma \cdot (\hat{\mathrm{p}} \times \hat{\tilde{\mathrm{p}}})\chi/2].$$

χ is the angle between $\hat{p}$ and $\hat{\tilde{p}}$. The new expectation value of $D^{(1)}(h)$ will be

$$< \mathrm{D}^{(1)}(\mathrm{h})_{0\mathrm{A}} >= \mathrm{D}^{(1)}(\mathrm{v}\tilde{\mathrm{u}})_{0\mathrm{A}},$$

which means $\tilde{u}$ cannot be chosen arbitrarily; v$\tilde{u}$ must have its axis in the XY plane. The new expectation value of $\tilde{\mathrm{E}}$ will have the same value as before, except replace u by v$\tilde{u}$.

All these theorems are relatively easy to prove: at an early step, use the similarity transformation to replace the poorly understood $\tilde{\mathrm{H}}$ by H; then use the extensive results of appendix C to approximate H. Note that the input unitary matrix $\tilde{u}$ no longer gives the peak value of the holonomy h.

Moving the parameters out of the XY plane seems to add nothing but complexity. If we stick with (u, $\vec{p}$), we already have enough parameters to tune the expectation values of both holonomy and triad. For $p_3 = 0$,

$$< \mathrm{D}^{(1)}(\mathrm{h})_{0\mathrm{A}} >= \mathrm{D}^{(1)}(\mathrm{u})_{0\mathrm{A}};$$

therefore we can adjust the holonomy to any value by adjusting u. There are only two adjustable parameters left, (p_1, p_2). However, the remaining expectation value, $< \tilde{\mathrm{E}}^x_A >$, must lie in a plane perpendicular to $D^{(1)}(u)_{0A}$. Therefore only two parameters are needed to tune its value.

IV V sqared, V, and V inverse

This paper is built on two approximations. The first approximation, an expansion in inverse powers of e^p, is used in appendix C to obtain a manageable expression for D(H).

The second approximation is an expansion in powers of $1/\sqrt{< L >}$: appendix D shows that small correction (SC) terms are down by factors of $1/\sqrt{< L >}$.

This latter expansion is not a traditional perturbative expansion: there is no Hamiltonian with an unperturbed part and known solutions, plus a small perturbing potential. However, there is a small parameter, $1/\sqrt{<L>}$; and perhaps more importantly, there is a complete set of orthonormal states. In this section I use this framework to compute the approximate square root and approximate inverse of the volume operator.

The next subsection needs a fact established in appendix D: the SC states are proportional to moments of the Gaussians in D(H). I. e. the original coherent state contains D(H), whereas the SC states contain D(H) times powers of M or (L - $<L>$). This is perhaps plausible because of the way the SC states are generated: by power-series expanding a summand in powers of M and (L - $<L>$).

A SC States as a Complete Subset

The expansion

$$\tilde{\mathrm{E}} \mid \mathrm{u}, \tilde{\mathrm{p}}\rangle = <\tilde{\mathrm{E}}> \mid \mathrm{u}, \tilde{\mathrm{p}}\rangle + \mathrm{SC}$$

resembles a perturbation expansion in certain respects. It is an expansion in a small parameter, $1/\sqrt{<L>}$, since SC terms are down by $1/\sqrt{<L>}$. Also, the states contained in the SC terms, plus the original state $\mid u, \vec{p}\rangle$, form a *complete* subset of the set of coherent states. The set of coherent states

$$\{\mid \mathrm{u}, \tilde{\mathrm{p}}\rangle, \forall\, \mathrm{u}, \tilde{\mathrm{p}}\}$$

is overcomplete, therefore not suitable for perturbation theory, which requires a complete set. The complete subset is therefore a more natural basis for calculations than the original set of coherent states.

The complete subset is constructed by repeated application of the $\tilde{\mathrm{E}}$ operators to the original coherent state and the states contained in the SC terms. Schematically,

$$\begin{aligned}
\tilde{\mathrm{E}} \mid u, \vec{p}\rangle &= <\tilde{\mathrm{E}}>\mid u, \vec{p}\rangle + \sum_X b_{1X} \mid m1(X)\rangle; \\
\tilde{\mathrm{E}} \mid m1(X)\rangle &= <\tilde{\mathrm{E}}>\mid m1(X)\rangle \\
&\quad + b \mid m2(M)\rangle + c \mid u, \vec{p}\rangle \\
&\quad + d \mid m1(M), m1(L)\rangle,
\end{aligned} \tag{30}$$

etc.. The ket $\mid m1(X)\rangle$ is the original ket, $\mid u, \vec{p}\rangle$, except that the Gaussian $D^{(L)}(H)_{M0}$ factors are replaced by

$$(\mathrm{X}- < \mathrm{X} >)\mathrm{D}^{(\mathrm{L})}(\mathrm{H})_{\mathrm{M0}},$$

X = L or M; i. e. the m1 stands for "first moment" of the Gaussians. Similarly, $\mid m2(X)\rangle$ is proportional to the second moment, $(X- < X >)^2$, etc.. (There is a similar series of SC states associated with the holonomy operator. In this section I consider only the series generated by the $\tilde{\mathrm{E}}$, because the operators discussed in this section depend only on $\tilde{\mathrm{E}}$.) The b_{1X} and b, c, d factors are down by $1/\sqrt{L}$; the terms containing these coefficients are small corrections.

The presence of a "c" term in eq. (30) is perhaps surprising: the $\tilde{\mathrm{E}}$ operator has the ability to *remove* one power of M as well as add one power. For a full discussion, see eq. (94) in appendix D.

Each state in the set

$$\{\mid \mathrm{mp(X), mq(Y)}\rangle, \forall \, \mathrm{integers\ p,q}\} \tag{31}$$

can be shown to be a linear combination of states in the original coherent basis. (This is clear from the overcomplete nature of the original basis; and see appendix D for an example.) Therefore the series forms a subset of the original basis.

This subset is complete and can be made orthogonal. On going from the original state to the SC states, one replaces the D(H) factor by D(H) times a power of $X- < X >$. Since the original factor D(H) is Gaussian in X, one is replacing the Gaussian by Gaussian times a Hermite polynomial, i.e. by a wavefunction for a simple harmonic oscillator (actually, two wavefunctions, one depending on M, and one depending on L, since D(H) is a Gaussian in M times a Gaussian in L). When the dot product of two SC states is computed, the factors depending on D(h) and D(u) always drop out. (Examples of this behavior are given in appendix D.) What remains is a sum over L and M, times products of simple harmonic oscillator wavefunctions in L and M. The sums over L and M can be turned into integrals, and the orthonormality follows.

The complete subset is a natural basis to use when computing the square root of $({}^{(2)}\mathrm{V})^2$ and the inverse of ${}^{(2)}\mathrm{V}$. In particular, ${}^{(2)}\mathrm{V}$ has a kernel, so that its inverse can only be approximate; the perturbation approach is useful in determining when the approximation breaks down.

B $({}^{(2)}\mathrm{V})^2$

This section calculates the action of the volume squared operator on the coherent states. This is the first point where I must take into account the fact that there is one holonomy for the x direction and another for the y direction.

$$\mid \mathrm{u}, \tilde{\mathrm{p}}\rangle \rightarrow \mid \mathrm{u_x}, \tilde{\mathrm{p}}_\mathrm{x}; \mathrm{u_y}, \tilde{\mathrm{p}}_\mathrm{y}\rangle = \mid \mathrm{u_x}, \tilde{\mathrm{p}}_\mathrm{x}\rangle \mid \mathrm{u_y}, \tilde{\mathrm{p}}_\mathrm{y}\rangle.$$

The volume operator contains both $\tilde{\mathrm{E}}^A_x$, which acts only on the first ket, and $\tilde{\mathrm{E}}^A_y$, which acts only on the second ket.

The $\tilde{\mathrm{E}}$ matrix is block-diagonal because of gauge choices [5]. The volume operator ${}^{(3)}\mathrm{V}$ therefore simplifies to

$$\begin{aligned} ({}^{(3)}\mathrm{V}\,)^2 &= \tilde{\mathrm{E}}^z_Z\, \epsilon_{ZAB}\, \tilde{\mathrm{E}}^x_A\, \tilde{\mathrm{E}}^y_B \\ &:= \tilde{\mathrm{E}}^z_Z\, {}^{(2)}\tilde{\mathrm{E}}\ . \end{aligned} \tag{32}$$

The holonomies along the z direction are eigenstates of $\tilde{\mathrm{E}}^z_Z$. Since $\tilde{\mathrm{E}}^z_Z$ is diagonal in the basis chosen, the non-trivial part of the volume operator is

$$({}^{(2)}\mathrm{V})^2 = {}^{(2)}\tilde{\mathrm{E}}\ , \tag{33}$$

which is the determinant of the transverse components of $\tilde{\mathrm{E}}$. In what follows I will refer to this (or its square root) as the "volume" operator, for short, even though ${}^{(2)}\mathrm{V}$ has the dimensions of an area. (I cannot call ${}^{(2)}\mathrm{V}$, e.g. transverse area; $\tilde{\mathrm{E}}^z_Z$ is the transverse area.)

The action of the $\tilde{\mathrm{E}}$ on the coherent states is given at eq. (88).

$$\begin{aligned} (\gamma\kappa/2)^{-2}({}^{(2)}\mathrm{V})^2\ \mid u_x, \vec{p_x}\rangle \mid u_y, \vec{p_y}\rangle &= A_{00} \mid u_x, \vec{p_x}\rangle \mid u_y, \vec{p_y}\rangle \\ &+ \sum_{Xy} A_{01X} \mid u_x, \vec{p_x}\rangle \mid m1(X_y)\rangle \\ &+ \sum_{Xx} A_{10X} \mid m1(X_x)\rangle \mid u_y, \vec{p_y}\rangle \\ &+ \text{order}\, 1/\sqrt{L_x^p L_y^q},\ \text{p+q} = 2. \end{aligned} \tag{34}$$

The lowest order SC states are m1 states (first moment states) such as $\mid m1(X_y)\rangle$ with $X_y = M_y$ or L_y. From eq. (88), the leading term

is order $L_x L_y$,

$$
\begin{aligned}
A_{00} &= (\gamma\kappa/2)^{-2} < \tilde{\mathrm{E}}^x_A >< \tilde{\mathrm{E}}^y_B > \epsilon_{ZAB} \\
&\sim\ < L_x >< L_y > \times \text{order unity}.
\end{aligned} \tag{35}
$$

From eqs. (89) and (103) the non-leading terms are suppressed by factors of $1/\sqrt{L}$.

$$
\mathrm{A_{mnX}} = \text{order} < \mathrm{L_x L_y} > (\sqrt{1/ < \mathrm{L_x} >})^{\mathrm{m}} (\sqrt{1/ < \mathrm{L_y} >})^{\mathrm{n}}. \tag{36}
$$

The action of $({}^{(2)}\mathrm{V})^2$ on other members of the complete set is similar, for example

$$
\begin{aligned}
(\gamma\kappa/2)^{-2} ({}^{(2)}\mathrm{V})^2 \mid m1(M_x)\rangle \mid u_y, \vec{p}_y\rangle = A_{00} \mid m1(M_x)\rangle \mid u_y, \vec{p}_y\rangle \\
+ \sum_{Xy} B_{01MX} \mid m1(M_x)\rangle \mid m1(X_y)\rangle \\
+ B_{10M} \mid m2(M)\rangle \mid u_y, \vec{p}_y\rangle \\
+ C_{10M} \mid u_x, \vec{p}_x\rangle \mid u_y, \vec{p}_y\rangle.
\end{aligned} \tag{37}
$$

The B_{mn} and C_{mn} are suppressed by the same factors as the A_{mn}, eq. (36). Eq. (37) implies that the other members of the complete set are also approximate eigenfunctions of $({}^{(2)}\mathrm{V})^2$ with the same eigenvalue (same leading coefficient A_{00}).

The above results suggest that ${}^{(2)}\mathrm{V}$, the square root of the volume operator, is also a double power series in $1/\sqrt{L_i}$, i = x,y . The series for $({}^{(2)}\mathrm{V})^2$, eqs. (34) and (37) have $L_x L_y$ as highest power, which suggests that the ${}^{(2)}\mathrm{V}$ series has highest power $\sqrt{L_x L_y}$. In the next subsection I propose an ansatz for ${}^{(2)}\mathrm{V}$ having this form.

C The Operator ${}^{(2)}\mathrm{V}$

The ansatz for ${}^{(2)}\mathrm{V}$ is

$$
\begin{aligned}
(\gamma\kappa/2)^{-1(2)}\mathrm{V} \ \mid u_x, \vec{p}_x\rangle \mid u_y, \vec{p}_y\rangle = a_{00} \mid u_x, \vec{p}_x\rangle \mid u_y, \vec{p}_y\rangle \\
+ \sum_{Xy} a_{01X} \mid u_x, \vec{p}_x\rangle \mid m1(X_y)\rangle \\
+ \sum_{Xx} a_{10X} \mid m1(X_x)\rangle \mid u_y, \vec{p}_y\rangle + \cdots ; \\
(\gamma\kappa/2)^{-1(2)}\mathrm{V} \ \mid m1(X_x)\rangle \mid u_y, \vec{p}_y\rangle = a_{00} \mid m1(X_x)\rangle \mid u_y, \vec{p}_y\rangle \\
+ c_{10X} \mid u_x, \vec{p}_x\rangle \mid u_y, \vec{p}_y\rangle + \cdots ; \\
(\gamma\kappa/2)^{-1(2)}\mathrm{V} \ \mid u_x, \vec{p}_x\rangle \mid m1(X_y\rangle = a_{00} \mid u_x, \vec{p}_x\rangle \mid m1(X_y)\rangle \\
+ c_{01X} \mid u_x, \vec{p}_x\rangle \mid u_y, \vec{p}_y\rangle + \cdots ,
\end{aligned} \tag{38}
$$

where

$$
\mathrm{a_{mn}, c_{mn} = order} \sqrt{\mathrm{L_x L_y}} (\sqrt{1/\mathrm{L_x}})^{\mathrm{m}} (\sqrt{1/\mathrm{L_y}})^{\mathrm{n}}. \tag{39}
$$

I wish to compute the a's, the coefficients giving the action of $\tilde{\mathrm{E}}$ on the original coherent state, to zeroth and first order. To do this I will need to know only the zeroth order term in the action of $\tilde{\mathrm{E}}$ on the higher moments; however, I will include the first order c_{mnX} terms to make the point that the diagonal elements

$$
\langle \mathrm{u_x, \tilde{p}_x} \mid \langle \mathrm{u_y, \tilde{p}_y} \mid {}^{(2)}\mathrm{V} \ \mid \mathrm{u_x, \tilde{p}_x} \rangle \mid \mathrm{u_y, \tilde{p}_y} \rangle
$$

have no first order corrections.

To determine the a's, I set

$$
({}^{(2)}\mathrm{V})^2 \ \mid \mathrm{u_x, \tilde{p}_x}\rangle \mid \mathrm{u_y, \tilde{p}_y}\rangle = {}^{(2)}\mathrm{V} \ \cdot {}^{(2)}\mathrm{V} \ \mid \mathrm{u_x, \tilde{p}_x}\rangle \mid \mathrm{u_y, \tilde{p}_y}\rangle. \tag{40}
$$

The left-hand side will be an expansion linear in A's; the right-hand side will be an expansion quadratic in a's. Both sides will be expansions in the orthonormal basis consisting of the original $\mid u_x, \vec{p}_x\rangle \mid u_y, \vec{p}_y\rangle$, plus states with at most one $\mid u_i, \vec{p}_i\rangle$ replaced by a $\mid m1(X)\rangle$. (States with one m2 ket, or both original kets replaced by m1 kets, do not contribute if we are keeping only zeroth plus first order.) As in standard perturbation theory, one can use orthonormality to project out the coefficient of each state. This yields a set of equations relating A's to expressions quadratic in a's.

The quadratic equations can be solved for the a's; phase ambiguities are resolved by demanding that ${}^{(2)}\mathrm{V}$ have positive eigenvalues.

Now write out both sides of eq. (40). The left-hand side of this equation was written out at eq. (34). From eq. (38), the right-hand side is

$$
\begin{aligned}
\text{rhs} &= {}^{(2)}\mathrm{V}\,[a_{00} \mid u_x, \vec{p}_x\rangle \mid u_y, \vec{p}_y\rangle \\
&\quad + \sum_{Xy} a_{01X} \mid u_x, \vec{p}_x\rangle \mid m1(X_y)\rangle \\
&\quad + \sum_{Xx} a_{10X} \mid m1(X_x)\rangle \mid u_y, \vec{p}_y\rangle + \cdots] \\
&= a_{00}\,[a_{00} \mid u_x, \vec{p}_x\rangle \mid u_y, \vec{p}_y\rangle \\
&\quad + \sum_{Xy} a_{01X} \mid u_x, \vec{p}_x\rangle \mid m1(X_y)\rangle \\
&\quad + \sum_{Xx} a_{10X} \mid m1(X_x)\rangle \mid u_y, \vec{p}_y\rangle + \cdots] \\
&\quad + \sum_{Xy} a_{01X}[a_{00} \mid u_x, \vec{p}_x\rangle \mid m1(X_y)\rangle \\
&\quad + c_{01X} \mid u_x, \vec{p}_x\rangle \mid u_y, \vec{p}_y\rangle + \cdots] \\
&\quad + \sum_{Xx} a_{10X}[a_{00} \mid m1(X_x)\rangle \mid u_y, \vec{p}_y\rangle \\
&\quad + c_{10X} \mid u_x, \vec{p}_x\rangle \mid u_y, \vec{p}_y\rangle + \cdots] \\
&\quad + \cdots .
\end{aligned} \tag{41}
$$

Set this expression equal to eq. (34) and invoke orthonormality to equate corresponding coefficients.

$$
\begin{aligned}
(\pm 1)A_{00} &= a_{00}^2 + \sum_X [a_{01X}c_{01X} + a_{10X}c_{10X}]; \\
(\pm 1)A_{01X} &= 2a_{00}a_{01X}; \\
(\pm 1)A_{10X} &= 2a_{00}a_{10X}.
\end{aligned} \tag{42}
$$

The $\pm 1 = \mathrm{sign}(A_{00})$ is inserted to insure that the quantity $(\pm 1)A_{00}$ is positive. The matrix needed for construction of the Hamiltonian is $\mid {}^{(2)}\mathrm{V} \mid$, not ${}^{(2)}\mathrm{V}$; i.e. $({}^{(2)}\mathrm{V})^2$ may have eigenvalues of either sign, but the eigenvalues of ${}^{(2)}\mathrm{V}$ must all be positive (or zero). Since the

diagonal, A_{00} element of the $({}^{(2)}\mathrm{V})^2$ operator is largest, the sign of the eigenvalue will be determined by the sign of A_{00}.

The solutions to eq. (42) are

$$\begin{aligned} a_{00} &= \sqrt{\pm A_{00}}; \\ a_{01X} &= \pm A_{01X}/(2\sqrt{\pm A_{00}}); \\ a_{10X} &= \pm A_{10X}/(2\sqrt{\pm A_{00}}). \end{aligned} \tag{43}$$

Note the sum on the first line of eq. (42) is second order and may be neglected; therefore all three lines of eq. (43) are already correct to first order. The first order corrections to a_{00} vanish.

D The Inverse of ${}^{(2)}\mathrm{V}$

The matrix ${}^{(2)}\mathrm{V}$ has a kernel, and cannot possibly have an exact inverse. However, if ${}^{(2)}\mathrm{V}$ acts on a coherent state which is an approximate eigenvector of ${}^{(2)}\mathrm{V}$, with very large eigenvalue, then one can neglect the states in the kernel, because ${}^{(2)}\mathrm{V}$ connects the original state only to other (coherent and mn(X)) states having large eigenvalue: note the structure of eq. (38).

In this situation it is possible to calculate an approximate inverse for ${}^{(2)}\mathrm{V}$. As in the calculation of ${}^{(2)}\mathrm{V}$, I work only to zero and first order in $1/\sqrt{L}$.

The matrix elements of ${}^{(2)}\mathrm{V}$ were assumed to be order $\sqrt{L_x L_y}$, times a power series in powers of $1/\sqrt{L_i}$. This suggests a corresponding ansatz for matrix elements of $({}^{(2)}\mathrm{V})^{-1}$: they are order $1/\sqrt{L_x L_y}$, times, again, a power series in powers of $1/\sqrt{L_i}$.

Also, the approximate inverse must obey

$$\delta(\mathrm{ix.jx})\delta(\mathrm{iy,jy}) = \Sigma_{\mathrm{nx,ny}}\langle \mathrm{jx,jy} \mid ({}^{(2)}\mathrm{V})^{-1} \mid \mathrm{nx,ny}\rangle\langle \mathrm{nx,ny} \mid {}^{(2)}\mathrm{V} \mid \mathrm{ix,iy}\rangle. \tag{44}$$

The indices (ix, iy), (jx, jy), (nx, ny) label states in the complete set, eq. (31).

As a first step I keep only the leading, zeroth order, contributions to the matrix ${}^{(2)}\mathrm{V}$. In zeroth order ${}^{(2)}\mathrm{V}$ is diagonal:

$$\langle \mathrm{nx,ny} \mid {}^{(2)}\mathrm{V} \mid \mathrm{ix,iy}\rangle = \delta(\mathrm{nx,ix})\delta(\mathrm{ny,iy})\mathrm{a}_{00}.$$

Therefore the matrix $({}^{(2)}\mathrm{V})^{-1}$ is also diagonal. To zeroth order,

$$\langle \mathrm{jx}, \mathrm{jy} \mid ({}^{(2)}\mathrm{V}\,)^{-1} \mid \mathrm{nx}, \mathrm{ny}\rangle = \delta(\mathrm{jx}, \mathrm{nx})\delta(\mathrm{jy}, \mathrm{ny})/\mathrm{a}_{00}. \tag{45}$$

Next, I include first order contributions to ${}^{(2)}\mathrm{V}$. In the sum over (nx,ny), eq. (44), I can drop all products of matrix elements which are second order or higher. Then the matrix elements in eq. (44) must be diagonal, with each ket index equal to the corresponding bra index (ix = nx, nx = jx, etc.), except for one pair, which I will call the off-diagonal pair. For example, suppose the off-diagonal pair is ix $\neq$ jx; then the requirement of at most one off-diagonal matrix element implies only two values of nx are possible; either ix = nx but jx $\neq$ nx; or the reverse, ix $\neq$ nx but jx = nx. Any other choice of nx would lead to a product of two first order matrix elements, therefore a contribution to the sum eq. (44) which is second order and negligible. Also, if the off diagonal pair is an x-pair (as in our example, where ix $\neq$ jx) then all y dependence must be diagonal to avoid second order contributions: iy = ny = jy.

For a specific example with ix $\neq$ jx, consider ix = the m1(M) state, jx = the original coherent state. Eq. (44) becomes

$$\begin{aligned} 0 &= \Sigma_{nx}\langle u_x, \vec{p}_x; \cdots \mid {}^{(2)}\mathrm{V}^{-1} \mid nx, \cdots\rangle\langle nx, \cdots \mid {}^{(2)}\mathrm{V} \mid m1(M_x); \cdots\rangle \\ &= \langle u_x, \vec{p}_x \mid {}^{(2)}\mathrm{V}^{-1} \mid u_x, \vec{p}_x\rangle\langle u_x, \vec{p}_x \mid {}^{(2)}\mathrm{V} \mid m1(M_x)\rangle \\ &\quad + \langle u_x, \vec{p}_x \mid {}^{(2)}\mathrm{V}^{-1} \mid m1(M_x)\rangle\langle m1(M_x) \mid {}^{(2)}\mathrm{V} \mid m1(M_x)\rangle. \end{aligned} \tag{46}$$

The $\cdots$ on the first line indicate suppressed labels iy, ny, jy which are all equal and do not change. The intermediate state $\mid nx\rangle = \mid m1(L_x)\rangle$ does not contribute; matrix elements

$$\langle \mathrm{m1(L)} \mid {}^{(2)}\mathrm{V} \mid \mathrm{m1(M)}\rangle$$

turn out to be second order.

Since three out of four of the matrix elements in eq. (46) are known to the required order, I can solve for the fourth matrix element.

$$\langle \mathrm{u_x}, \tilde{\mathrm{p}}_\mathrm{x}; \mathrm{iy} \mid {}^{(2)}\mathrm{V}^{-1} \mid \mathrm{m1(M_x)}; \mathrm{iy}\rangle = -\mathrm{a}^*_{10\mathrm{M}}/(\mathrm{a}_{00})^2. \tag{47}$$

By systematically considering all $ix \neq jx$ and $iy \neq jy$ pairs, I can determine all first order elements of the inverse. For example,

$$\langle ix; u_y, \vec{p}_y \mid {}^{(2)}\mathrm{V}^{-1} \mid ix; m1(M_y)\rangle \;\; = \;\; -a^*_{01M}/(a_{00})^2, \tag{48}$$

which will be recognized as a relabeled version of eq. (47).

Eq. (45) for the diagonal elements of $({}^{(2)}\mathrm{V}\,)^{-1}$ is already correct to first order; there are no first order corrections to the diagonal elements. Proof: the diagonal elements have ix = jx and iy = jy. If also ix = nx and iy = ny, then all matrix elements are diagonal and we need a first order correction to a diagonal element. There are none (see eq. (43)). If (say) ix $\neq$ nx, then jx $\neq$ nx also. This contribution to the sum over n_x is second order, and may be dropped from the sum. Formula eq. (45) for the diagonal elements of $({}^{(2)}\mathrm{V}\,)^{-1}$ is correct to first order. $\square$

${}^{(2)}\mathrm{V}$ can have only an approximate inverse. We can estimate the values of the parameters when the series begins to diverge and ${}^{(2)}\mathrm{V}$ no longer has an approximate inverse, by examining the first order corrections to ${}^{(2)}\mathrm{V}$. All matrix elements of $({}^{(2)}\mathrm{V}\,)^{-1}$ have at least one factor of a_{00} in the denominator, where a_{00} is the ${}^{(2)}\mathrm{V}$ eigenvector for the original coherent state. This is as expected: if the volume eigenvector is small (if the original state is close to the kernel), then the inverse does not exist. To get some measure of closeness to the kernel, consider the first order terms. They are down by factors of $(a^*_{01} \text{ or } a^*_{10})/a_{00}$. These expressions are small, of order $1/\sqrt{< L >}$, *unless* a_{00} is small, of order $\sqrt{< L >}$ rather than $\sqrt{< L_x L_y >}$. a_{00} is order $\sqrt{< L_x L_y \,\mathrm{sine} >}$, where sine denotes the sine of the angle between the two angular momenta. When this sine becomes small, of order $1/< L >$, ${}^{(2)}\mathrm{V}$ loses its inverse.

E The square root of $\tilde{\mathrm{E}}^z_Z$

At the beginning of this section I stated that the square of the volume operator is a product of $\tilde{\mathrm{E}}^z_Z$ times $({}^{(2)}\mathrm{V})^2$; and the resultant $\sqrt{\tilde{\mathrm{E}}^z_Z}$ factor was trivial because the basis states are eigenfunctions of $\tilde{\mathrm{E}}^z_Z$. This is an oversimplification. The literature contains at least two recipes for extracting the square root of volume operators. At present it does not seem possible to distinguish between the two approaches, using general principles; and I need to state which approach I am using here.

When more than three edges terminate at a given vertex, the $(\mathrm{volume})^2$ operator grasps each triplet of edges in turn, generating a series of amplitudes, one for each triplet. To parallel a terminology from classical optics: should one add first, then take the square

root? Or take the square root first, then add? I.e. should one add the amplitudes from each triplet, then take the square root of the magnitude of the result; or should one take the square root of the magnitude of each amplitude first, then add. The literature contains advocates for both "add first" [6, 7] and "take the square root first" [8, 9] choices. For a discussion of the distinct regularization schemes leading to each choice see reference [7]. Presumably no final choice between the schemes can be made until the two choices have been checked in applications.

In this paper I adopt the "add first" choice, for the following (non-rigorous) reason. Consider a four-valent vertex v_n such that two of the edges meeting at the vertex are tangent, one ingoing and one outgoing. Rotate the gauge at the vertex so that the holonomies along these edges are pure S_Z, with S_Z eigenvalues M_f and M_i for outgoing and ingoing vertices, respectively. The vertex has two holonomies,

$$\exp[\mathrm{i}\int_{\mathrm{n}}^{\mathrm{n+1}} \mathrm{M_f}\, \mathrm{A}_i^Z \mathrm{dx^i}\,];\ \exp[\mathrm{i}\int_{\mathrm{n-1}}^{\mathrm{n}} \mathrm{M_i}\, \mathrm{A}_i^Z \mathrm{dx^i}\,].$$

In the "add first" prescription, the contributions to (volume)^2 from these two holonomies are added, giving a factor of $(M_f + M_i)/2$. The factor of $1/2$ comes from the integrations over half a delta function. The volume is proportional to the square root

$$\sqrt{\mid \mathrm{M_f} + \mathrm{M_i} \mid /2}.$$

In the "square root first" prescription, the corresponding factor would be

$$\sqrt{\mid \mathrm{M_i} \mid /2} + \sqrt{\mid \mathrm{M_f} \mid /2}.$$

In the special case $M_f = M_i$, it is possible to view these two holonomies as a single holonomy passing through the vertex v_n. When this is grasped by the (volume)^2 the contribution is proportional to M_i with no $1/2$, and the volume is proportional to $\sqrt{\mid M_i \mid}$. This result for one holonomy equals the $M_f = M_i$ limit of the result for two different holonomies, only if the "add first" prescription is used.

The planar case resembles the example just given. Each vertex has two holonomies,

$$\exp[\mathrm{i}\int_{\mathrm{n}}^{\mathrm{n+1}} \mathrm{M_f A}_z^Z \mathrm{dz}\,];\exp[\mathrm{i}\int_{\mathrm{n-1}}^{\mathrm{n}} \mathrm{M_i A}_z^Z \mathrm{dz}\,].$$

Given the "add first" choice, the eigenvalues of $\tilde{\mathrm{E}}^z_Z$ from the two holonomies should be added, yielding the eigenvalue

$$(\gamma\kappa/2)(\mathrm{M_f} + \mathrm{M_i}). \tag{49}$$

The eigenvalue of the volume operator contains the square root of the magnitude of the above expression.

V The commutator [h,V]

At this point it might seem we are finished. The basic building blocks for the Hamiltonian are the holonomy and the volume operators; formulas of the previous sections should be adequate to evaluate most matrix elements of $h^{(1/2)}$ and ${}^{(3)}\mathrm{V}$.

There is one exception, however. Commutators such as

$$\begin{aligned}\langle u,\vec{p} \mid [h^{(1/2)}, {}^{(3)}\mathrm{V}\,] \mid u,\vec{p}\rangle &= \langle u,\vec{p} \mid h^{(1/2)} \mid n\rangle\langle n \mid {}^{(3)}\mathrm{V} \mid u,\vec{p}\rangle \\ &\quad -\langle u,\vec{p} \mid {}^{(3)}\mathrm{V} \mid n\rangle\langle n \mid h^{(1/2)} \mid u,\vec{p}\rangle\end{aligned}$$

are a small difference between two large terms. The contribution from the leading term, $\mid n\rangle = \mid u,\vec{p}\rangle$, cancels out of the difference, and it is very hard to evaluate this commutator using only the results of previous sections.

Thiemann and Winkler have shown that, in the classical limit, the quantum mechanical commutator equals the classical Poisson bracket [4].

$$\langle \mathrm{u},\tilde{\mathrm{p}} \mid [\mathrm{O}_1,\mathrm{O}_2] \mid \mathrm{u},\tilde{\mathrm{p}}\rangle/\mathrm{i}\hbar = \{\mathrm{O}_1,\mathrm{O}_2\}(\mathrm{u},\tilde{\mathrm{p}}) \tag{50}$$

On the right, one computes the Poisson bracket, treating the operators as classical fields, then evaluates the result at the peak values for the coherent state. Since the Poisson bracket is no longer a small difference between two large terms in general, it can be evaluated using the formulas of previous sections.

[V,h] commutators occur (for example) in the Euclidean part of the Hamiltonian, which contains the following terms (in a field theoretic formulation, before discretization on a spin network)

$$\begin{aligned}&F^Z_{xy}\,{}^{(2)}\tilde{\mathrm{E}}\;/e^{(3)};\\ &F^A_{zx}\,\tilde{\mathrm{E}}^z_Z\,\tilde{\mathrm{E}}^x_B\,\epsilon_{ZBA}/e^{(3)};\\ &F^A_{zy}\,\tilde{\mathrm{E}}^z_Z\,\tilde{\mathrm{E}}^y_B\,\epsilon_{ZBA}/e^{(3)}.\end{aligned} \tag{51}$$

In this section A,B = X,Y only. In spin network theory for the planar case, these terms become commutators

$$\begin{aligned} &2F^Z_{xy}\,Tr\{\sigma_Z\,h_z[h_z^{-1},{}^{(3)}\mathrm{V}\,]\}; \\ &2F^A_{zx}\,Tr\{\sigma_A\,h_y[h_y^{-1},{}^{(3)}\mathrm{V}\,]\}; \\ &2F^A_{zy}\,Tr\{\sigma_A\,h_x[h_x^{-1},{}^{(3)}\mathrm{V}\,]\}. \end{aligned} \tag{52}$$

h is short for $h^{(1/2)}$.

Eq. (52) is not yet in its final spin network form. Each F_{ij} must be replaced by a product of four holonomies encircling the four sides of the area ij. This is not the place to discuss the construction of these holonomy products; however, one issue does need to be addressed. In general a product of four spin 1/2 holonomies will contain all the Pauli matrices. For instance, the holonomies replacing $F^Z_{xy}\,\sigma_Z$ could contain σ_A matrices, A $\neq$ Z, or even the 2x2 unit matrix. Fortunately, there exist several possible holonomy products which reduce to the same F_{ij} in a field theory limit. (For example, proceed counterclockwise *vs.* clockwise around the area.) One can choose a linear combination of the possibilities which is pure σ_Z or pure σ_A, as needed to reproduce the classical results. In what follows I will assume this has been done, so that the only traces required are those shown in eq. (52).

Also, one must decide how to choose the holonomies h_k in eq. (52). In spin network theory the Euclidean Hamiltonian has the following form.

$$\mathrm{H_E} \propto \epsilon^{\mathrm{ijk}}\,\mathrm{h_{ij}}\,\mathrm{h_k}\,[\mathrm{V},\mathrm{h_k^{-1}}],$$

where h_{ij} stands for the product of four holonomies which replaces the F_{ij} of eq. (52). When this Hamiltonian acts on a spin network, the h_k and h_{ij} are taken to lie along edges of the network. This requirement removes some, but not all of the ambiguity. If the vertex is n-valent, there could be as many as n-2 edges which do not lie in the ij plane; and any one of these (n-2) edges could qualify as support for h_k. (In the planar case this ambiguity occurs for ij = xy, k = z, since at each vertex there are two z holonomies, one entering and one leaving.) I adopt the simplest choice for resolving this ambiguity, which is to average over the n-2 possibilities [10]. When combining the n-2 amplitudes contributed by each h_k, I adopt the same "add first" prescription as was used for the volume operator.

In the planar case it will be possible to evaluate the $[h_z^{-1}, {}^{(3)}\mathrm{V}\,]$ commutator exactly, since h_z is an eigenfunction of ${}^{(3)}\mathrm{V}$; see the next subsection. One can then evaluate this exact quantum expression in the classical, large quantum number limit, and compare the limit to the result obtained from eq. (50), or equivalently compare the limit to the classical field theoretic formula for the commutator. The comparison will not support any particular prescription for choosing the h_k, but does clarify some issues regarding eq. (50).

A Holonomies along z

As indicated just above, the h_z commutator can be done exactly, without using the theorem, eq. (50). The nth vertex has two holonomies directed along z, one incoming and one outgoing. Therefore the wavefunction at vertex n contains the product

$$\begin{aligned} h(z,n) &:= \exp[\, i \int_{n-1}^{n} \mathrm{A}_z^Z\, S_Z\, dz\,]\, \exp[\, i \int_{n}^{n+1} \mathrm{A}_z^Z\, S_Z\, dz\,] \\ &= \exp[\, i \int_{n-1}^{n} \mathrm{A}_z^Z\, M_i\, dz\,]\, \exp[\, i \int_{n}^{n+1} \mathrm{A}_z^Z\, M_f\, dz\,]. \quad (53) \end{aligned}$$

The total wavefunction is h(z,n)$|\, u_x, \vec{p}_x; u_y, \vec{p}_y\rangle$. When $h_z\,[h_z^{-1}, {}^{(3)}\mathrm{V}\,]$ acts on this wavefunction, the $\sqrt{|\ {}^{(2)}\tilde{\mathrm{E}}\ |}$ factor in ${}^{(3)}\mathrm{V} = \sqrt{|\ \tilde{\mathrm{E}}_Z^z\, {}^{(2)}\tilde{\mathrm{E}}\ |}$ merely factors out and acts on the coherent state factor. The non-trivial part of the commutator is determined by the $\sqrt{|\ \tilde{\mathrm{E}}_Z^z\ |}$ factor in the commutator acting on the h_z^{-1} factor in the commutator and the h(z,n) factor in the state.

It is straightforward to compute the action of $\sqrt{|\ \tilde{\mathrm{E}}_Z^z\ |}$ on h(z,n) and h_z^{-1}, since the operator $\tilde{\mathrm{E}}_Z^z$ is diagonal in the holonomy representation. We need only compute the action of $\tilde{\mathrm{E}}_Z^z$, add (or average) amplitudes appropriately, and then take a square root. The holonomy h_z^{-1} may be either incoming or outgoing. If it is outgoing, then

$$\begin{aligned} &[h_z^{-1}, \tilde{\mathrm{E}}_Z^z]\, h(z,n) \\ &\qquad = [\exp\{-i \int_{n}^{n+1} \mathrm{A}_z^Z (\pm 1/2)\, dz\}, \tilde{\mathrm{E}}_Z^z]\, h(z,n) \\ &\qquad = (\gamma\kappa/2) h^{-1} h(z,n) [(M_f + M_i) - (M_f + M_i \mp 1/2)]. \quad (54) \end{aligned}$$

The $\pm$ refers to the $[\pm, \pm]$ diagonal element of the matrix h_z^{-1}. The ingoing holonomy gives an identical contribution. This tells us immediately that the exact calculation throws no light on how to average over the various possible h_k. We get the same result, whether we use ingoing only, use outgoing only, or take the average.

I have chosen the prescription which averages incoming and outgoing contributions, then takes the square root of the magnitude. Note however the final two parentheses in eq. (54) should not be combined or averaged; they come from two separate applications of the volume operator.

$$\begin{aligned}
[h_z^{-1}, \sqrt{\eta \tilde{\mathrm{E}}_Z^z}]h(z,n) &= \sqrt{(\gamma\kappa/2)}h_z^{-1}h(z,n)[\sqrt{\mid M_f + M_i \mid} \\
&\quad -\sqrt{\mid M_f + M_i \mid \mp \eta/2}], \qquad (55)
\end{aligned}$$

where $\mid \tilde{\mathrm{E}}_Z^z \mid = \eta \tilde{\mathrm{E}}_Z^z$ and $\eta = \pm 1$ is the phase of the eigenvalue $M_f + M_i$.

From eq. (52) the result eq. (55) should be inserted into the trace

$$\begin{aligned}
T(Z,z)\,h(z,n) &:= Tr\{\sigma_Z h_z [h_z^{-1}, \sqrt{\eta \tilde{\mathrm{E}}_Z^z}]\,\}\,h(z,n) \\
&= \Sigma_{\pm}\{(\pm 1)\sqrt{(\gamma\kappa/2)}[\sqrt{\mid M_f + M_i \mid} \\
&\quad -\sqrt{\mid M_f + M_i \mid \mp \eta/2}]\,\}\,h(z,n) \\
&= \sqrt{(\gamma\kappa/2)}[-\sqrt{\mid M_f + M_i \mid - \eta/2} \\
&\quad +\sqrt{\mid M_f + M_i \mid + \eta/2}]\,h(z,n). \qquad (56)
\end{aligned}$$

Eq. (56) is an exact result. In the limit of large quantum numbers $\mid M_f + M_i \mid \gg 1/2$, the radicals can be expanded, yielding

$$\mathrm{T(Z,z)} \rightarrow \sqrt{(\gamma\kappa/2)}[\eta/(2\sqrt{\mid \mathrm{M_f} + \mathrm{M_i} \mid})\,]. \qquad (57)$$

This approximate result should be compared to the result from clas-

sical field theory (or from the theorem, eq. (50))

$$\begin{aligned} T(Z,z)h(z,n) &\rightarrow Tr\{\sigma_Z(1)[1 - i\mathrm{A}_z^Z S_Z dz, \sqrt{\eta\tilde{\mathrm{E}}_Z^z}]\,\}h(z,n) \\ &= Tr\{\sigma_Z(1)[S_Z dz(\delta/\delta\tilde{\mathrm{E}}_Z^z)\sqrt{\eta\tilde{\mathrm{E}}_Z^z}]\,\}h(z,n) \\ &= (\eta/2)(\gamma\kappa/2)[1/\sqrt{\eta\tilde{\mathrm{E}}_Z^z}\,]\,h(z,n) \\ &= (\eta/2)\sqrt{(\gamma\kappa/(2\mid M_f + M_i \mid)}\,h(z,n). \qquad (58) \end{aligned}$$

This result agrees with the large quantum number limit, eq. (57).

This agreement sheds no light on the procedure for combining amplitudes, but does shed light on the theorem, eq. (50). The $\sqrt{\eta\tilde{\mathrm{E}}_Z^z}$ operator is not distributive. When acting on a product of two functions of holonomies,

$$\sqrt{\eta\tilde{\mathrm{E}}_Z^z}\,[\mathrm{f}_1(\mathrm{h})\mathrm{f}_2(\mathrm{h})] \neq \mathrm{f}_2(\mathrm{h})\sqrt{\eta\tilde{\mathrm{E}}_Z^z}\,[\mathrm{f}_1(\mathrm{h})] + \mathrm{f}_1(\mathrm{h})\sqrt{\eta\tilde{\mathrm{E}}_Z^z}\,[\mathrm{f}_2(\mathrm{h})].$$

This can be checked readily by choosing the f_i to be eigenvectors of $\tilde{\mathrm{E}}_Z^z$. In eq. (50) suppose one takes $O_1 = \sqrt{\eta\,\tilde{\mathrm{E}}_Z^z}$, $O_2 = f_1$. The theorem seems to state that the action of O_1 on f_1 is independent of the state on which the commutator acts (is independent of f_2). One might be tempted to question this result, given the non-distributive character of $\sqrt{\eta\,\tilde{\mathrm{E}}_Z^z}$. However, the classical limit of the exact quantum result agrees with the theorem.

When the eigenvalues $\mid M_i \mid$ of $\mid \tilde{\mathrm{E}}_Z^z \mid$ are large, naively one might expect $[h_z^{-1}, \sqrt{\mid \tilde{\mathrm{E}}_Z^z \mid}\,]$ to be order the eigenvalue of $\sqrt{\mid \tilde{\mathrm{E}}_Z^z \mid}$, $\sqrt{\mid M_i + M_f \mid}$; but instead it is of order $1/(2\sqrt{\mid M_f + M_i \mid})$, which is the derivative of $\sqrt{\mid M_i + M_f \mid}$. In the next subsection we will find that this commutator = derivative structure holds also for the commutators of transverse holonomies with the volume.

B Holonomies along x,y

Now consider commutators involving h_i, i = x,y. As before, ${}^{(3)}$V factors into ${}^{(2)}$V times $\sqrt{\mid \tilde{\mathrm{E}}_Z^z \mid}$; but now the $\sqrt{\mid \tilde{\mathrm{E}}_Z^z \mid}$ factors out. When acting on the h(z,n) factor in the wavefunction, it gives a factor of

$\sqrt{\mid M_f + M_i \mid}$, as at eq. (49). We are left with the following trace, which acts on the coherent ket.

$$\mathrm{T(A,i) = Trace}(\sigma_{\mathrm{A}}\, \mathrm{h_i}\, [\mathrm{h_i^{-1}}, {}^{(2)}\mathrm{V}\,]\,). \tag{59}$$

Since the case i = y may be obtained from results for i = x by a relabeling, it suffices to compute this trace for i = x.

The theorem, eq. (50), gives for the commutator in T(A,i)

$$\begin{aligned}
[h_x^{-1}, {}^{(2)}\mathrm{V}\,] &= i[\delta(h_x^{-1})/\delta \mathrm{A}_x^B][\delta({}^{(2)}\mathrm{V}\,)\delta\, \tilde{\mathrm{E}}_B^x](\gamma\kappa) \\
&= (\gamma\kappa/2)[S_B, h_x^{-1})]_+[\zeta\, \tilde{\mathrm{E}}_C^y\, \epsilon^{BC}/2{}^{(2)}\mathrm{V}\,],
\end{aligned} \tag{60}$$

where ${}^{(2)}\mathrm{V}$ is $\sqrt{\zeta\, \tilde{\mathrm{E}}_B^x\, \tilde{\mathrm{E}}_C^y\, \epsilon^{BC}}$ and $\zeta = \pm 1$ is the sign of the eigenvalue of $({}^{(2)}\mathrm{V})^2$. The anticommutator arises on the second line because the holonomy is a loop, beginning and ending at the vertex where the $\tilde{\mathrm{E}}$ acts. Therefore the grasp occurs on the far left and right of the holonomy.

The ${}^{(2)}\mathrm{V}$ in the denominator of the formula eq. (60) requires some discussion. Even though ${}^{(2)}\mathrm{V}$ has vanishing eigenvalues, there is no divergence. The ${}^{(2)}\mathrm{V}$ operator will be acting on coherent kets which are approximate eigenvectors of ${}^{(2)}\mathrm{V}$ with large eigenvalue. For this situation, from the discussion of $({}^{(2)}\mathrm{V}\,)^{-1}$ in section IV, the zero eigenvalues may be neglected. ${}^{(2)}\mathrm{V}$ connects the original large eigenvalue state primarily to other large eigenvalue states, rather than states in the kernel.

The ${}^{(2)}\mathrm{V}$ factor also does not give rise to a factor ordering ambiguity, even though it does not commute with the h_x^{-1}. The difference between two orderings equals a commutator, which is small.

$$\begin{aligned}
h_x(1/{}^{(2)}\mathrm{V}\,) &= (1/{}^{(2)}\mathrm{V}\,)\, h_x + [h_x, 1/{}^{(2)}\mathrm{V}\,]; \\
[h_x, 1/{}^{(2)}\mathrm{V}\,] &= (-1/({}^{(2)}\mathrm{V})^2\,)[h_x, {}^{(2)}\mathrm{V}\,]\, h_x \\
&= \mathrm{order}\,(h_x/{}^{(2)}\mathrm{V}\,)(\,[h_x, {}^{(2)}\mathrm{V}\,]/{}^{(2)}\mathrm{V}\,).
\end{aligned} \tag{61}$$

On the second line, I assume h_x is outgoing, so that the ${}^{(2)}\mathrm{V}$ operator overlaps with h_x on the left side of h_x. An ingoing holonomy would give the same final order of magnitude. In the present case, as in the previous subsection, the commutator resembles a derivative (now a derivative with respect to L rather than M) in that the commutator lowers the power of L by one. To see this, note ${}^{(2)}\mathrm{V}$ is order $\sqrt{L_x L_y}$, from the two $\tilde{\mathrm{E}}$ operators in ${}^{(2)}\mathrm{V}$. From eq. (60), $[h_x, {}^{(2)}\mathrm{V}\,]$ is order

$L_y/\sqrt{L_x L_y}$, down by a derivative with respect to L_x. (The S_B is just a Pauli matrix over 2; it is not order L.) Therefore the two orderings in eq. (61) differ by a factor which is suppressed by $[h_x, {}^{(2)}\mathrm{V}\,]/{}^{(2)}\mathrm{V}$ = order $1/L_x$. In the classical limit, factor ordering is not a problem.

I now substitute eq. (60) into the expression for the trace, eq. (59).

$$\begin{aligned}
T(A,x) &= Tr[\sigma_A(\sigma_B + h_x\sigma_B h_x^{-1})](1/2)(\gamma\kappa/2)\zeta\epsilon_{BC}\tilde{\mathrm{E}}^y_C/{}^{(2)}\mathrm{V} \\
&= Tr[\sigma_A(\sigma_B + \sigma_D D^{(1)}(h_x)_{DB})](1/2)(\gamma\kappa/2)\zeta\epsilon_{BC}\tilde{\mathrm{E}}^y_C/{}^{(2)}\mathrm{V} \\
&= [\delta_{AB} + D^{(1)}(h_x)_{AB}](\gamma\kappa/2)\,\zeta\,\epsilon_{BC}\,\tilde{\mathrm{E}}^y_C/{}^{(2)}\mathrm{V}\,; \\
D^{(1)}(h) &= D^{(1)}(-\phi+\pi/2, \theta, \phi-\pi/2). \qquad (62)
\end{aligned}$$

The second line uses the rotation property of the su(2) generators, eq. (65). The matrix on the last line is a full-angle rotation, not a half-angle rotation; we are on the SU(2) side of the SU(2) vs. O(3) divide discussed in section II of paper 1.

Eq. (62) is correct for T(A,x). For T(A,y) replace $h_x \rightarrow h_y$ and $\epsilon_{BC}\,\tilde{\mathrm{E}}^y_C \rightarrow \epsilon_{CB}\,\tilde{\mathrm{E}}^x_C$.

A note on commutator = derivative: taking the commutator does reduce the magnitude in the same way as taking a derivative; but the commutator does not preserve directions. Compare matrix elements of ${}^{(2)}\mathrm{V}$ to those of $[h_x^{-1}, {}^{(2)}\mathrm{V}\,]$, eq. (60). I replace the SU(2) objects $h^{(1/2)}$ by 0(3) objects $D(1)_{0A}$ to facilitate order of magnitude comparisons:

$$\begin{aligned}
< {}^{(2)}\mathrm{V} > &\sim \sqrt{<\tilde{\mathrm{E}}^x_M><\tilde{\mathrm{E}}^y_N>\epsilon_{MN}}; \\
< [D(1)_{0A}, {}^{(2)}\mathrm{V}\,] > &\sim \; < D(1)_{0B} >< \tilde{\mathrm{E}}^y_N > \epsilon_{MN}/ < {}^{(2)}\mathrm{V} > .
\end{aligned}$$

The net effect is to replace $<\tilde{\mathrm{E}}^x_M>$ by $< D(1)_{0B} >$. From eq. (9), $<\tilde{\mathrm{E}}^x_M>$ is order $< L_x >$

$$<\tilde{\mathrm{E}}^x_M> = <\mathrm{L_x}> (\hat{\mathrm{n}}_\mathrm{M}\cos\mu - \hat{\mathrm{n}}\times\hat{\mathrm{V}})_\mathrm{M}\sin\mu),$$

while its replacement is order unity. Evidently the commutator reduces magnitude in the same manner as a derivative.

However, $<\tilde{\mathrm{E}}^x_M>$ also contains a unit vector, and information about this vector has been lost. Presumably the commutator = derivative relation was exact for commutators with $\sqrt{\tilde{\mathrm{E}}^z_Z}$, because only one direction is involved. For other operators, the relation is useful when one is counting powers of L.

VI Conclusion

One technique used the present, planar calculation depends only on angular momentum theory, therefore applies also to the general SU(2) case (or to any symmetry which has recoupling coefficients analogous to 3J symbols). Given a coherent state with holonomy peaked at a matrix u, all peak values should be functions of u. Since the basic planar operators are spin 1, some step in the calculation should produce low-spin representations $D^{(1)}$ of u. In this paper I have used the formulas for the rotation behavior of Clebsch-Gordan coefficients, appendix A, to reveal this u dependence . The basic operators in the general SU(2) case are spin 1/2; therefore the same formulas will produce low-spin representations $D^{(1/2)}$ of u.

Those formulas also lead to a clean separation between leading terms and the small correction terms. Study of the SC terms reveals that they emphasize small fluctuations, values of the basic variables M and L which are near, but not at the mean values $< M >$ and $< L >$. Further, the standard deviation parameter t cannot be taken too different from $1/< L >$, or these SC terms will become large.

The coherent state approach is not like perturbation theory. One cannot improve on the initial approximation by taking more terms, essentially because perturbation theory uses a complete set, whereas coherent states are overcomplete.

Can one construct a complete subset of the overcomplete set, a subset suitable for perturbation theory? In effect, I did this in section IV. The subset consists of the original coherent state, which has Gaussian distributions in L and M, plus SC states with distributions given by moments of the Gaussians. The subset was used to construct a perturbation expansion for the inverse of ${}^{(2)}\mathrm{V}$. Indeed, all the formulas giving the action of the holonomy and $\tilde{\mathrm{E}}$ operators on the coherent state may be considered as perturbation expansions in this complete subset.

The perturbation calculation in section IV certainly is helpful. Using the perturbation expansion, one can put quantitative limits on when ${}^{(2)}\mathrm{V}$ no longer possesses an approximate inverse.

Also, the technique works quite generally. One can use moments to construct a complete subset, for calculations involving any kind of coherent state.

Is this perturbation approach useful for extending coherent state calculations to smaller values of L? Since higher terms in the perturbation series fall off as $1/\sqrt{L}$, perhaps one could extend the calculation to low values of L, L < 100, by including higher moment terms in the series.

However, there is a possible problem with spreading of the coherent wave packet. Coherent states are useful because their eigenvalues are strongly peaked at one central value. If this peak broadens rapidly in time, the states lose their coherent character. In a follow-on paper, I estimate the rate of spreading of a coherent state, and for small L the rate of spreading appears to be quite rapid [2]. The perturbation expansion is certainly useful in many contexts, but it may not be well suited for reaching small values of L.

Future work should choose a specific Hamiltonian, then check for wave propagation and correct commutation relations in the coherent state limit. Although I have not written down a specific Hamiltonian in this series of papers, I have listed constraints which limit the form of the Hamiltonian. (See for example the remarks following eq. (52).)

A Identities Involving D and S

For the convenience of the reader, this appendix includes brief derivations of two well-known identities involving the rotation matrices D and generators S. Throughout, I do not use the relation $D^{-1} = D\dagger$, so that the formulas in this appendix remain valid for matrices in SL(2,C). My conventions for Clebsch-Gordan coefficients and rotation matrices are those of Edmonds [11]

The first identity relates matrix elements of S to a Clebsch-Gordan coefficient.

$$\langle \mathrm{L}, \mathrm{M}' \mid \mathrm{S_A} \mid \mathrm{L}, \mathrm{M} \rangle = \sqrt{\mathrm{L}(\mathrm{L}+1)} \langle \mathrm{L}, \mathrm{M}' \mid \mathrm{L}, \mathrm{M}; 1, \mathrm{A} \rangle, \tag{63}$$

where the Clebsch-Gordan coefficients for $L1 \bigotimes L2 = L3$ are written in a braket notation as $\langle L3, M3 \mid L1, M1; L2, M2 \rangle$. The M dependence of the right hand side of eq. (63) is required by the Wigner-Eckart theorem applied to a spin one operator. To check the scalar coefficient $\sqrt{L(L+1)}$, square both sides and sum over M and A.

$$\langle \mathrm{L}, \mathrm{M}' \mid \tilde{\mathrm{S}}^2 \mid \mathrm{L}, \mathrm{M}' \rangle = \mathrm{L}(\mathrm{L}+1)\langle \mathrm{L}, \mathrm{M}' \mid \mathrm{L}, \mathrm{M}' \rangle.$$

The phase of the scalar coefficient can be verified by checking a simple example.

The second identity is the formula for reducing a product of two rotation matrices. (I suppress the labels L_i, which are obvious from context.)

$$\begin{aligned} D_{N1M1}D_{N2M2} &= \sum_{L3} \langle N3 \mid N1N2 \rangle D_{N3M3} \langle M3 \mid M1M2 \rangle; \\ \langle N3 \mid N1N2 \rangle D_{N1M1} &= D_{N3M3} \langle M3 \mid M1M2 \rangle D^{-1}_{M2N2}. \end{aligned} \tag{64}$$

The second line is a rewritten version of the first.

As an illustration of these rules, I obtain the usual rotation property for the S_A vector operator. Insert eq. (63) for S into the second line of eq. (64), and restore the L's.

$$\begin{aligned} \langle L, N3 \mid S_A \mid L, N1 \rangle D^{(L)}_{N1M1} &= D^{(L)}_{N3M3} \langle L, M3 \mid S_B \mid L, M1 \rangle D^{(1)-1}_{BA} \\ &\Leftrightarrow \\ \mathbf{S}_A \mathbf{D}^{(L)} &= \mathbf{D}^{(L)} \mathbf{S}_B D^{(1)-1}_{BA}. \end{aligned} \tag{65}$$

The last line uses a matrix notation to hide some indices. This formula is also valid for matrices D in SL(2,C).

For components of vectors, this paper use both Cartesian (X,Y,Z or 1,2,3) and spherical (+1,-1,0) indices. Strictly speaking, the two should be treated slightly differently when forming dot products.

$$\begin{aligned} \vec{A} \cdot \vec{B} &= A_X B_X + A_Y B_Y + A_Z B_Z \\ &= (A_-)^* B_- + (A_+)^* B_+ + A_3 B_3. \end{aligned} \tag{66}$$

Since spherical components are complex, the second dot product resembles the dot product in SU(3) rather than O(3). For the most part I have omitted the complex conjugation stars, trusting to the reader to know when to put them in. Note in most (but not all) cases, no star is needed because one of the indices refers to a final state, which can be considered starred. For example, there should be a star on one of the factors in eq. (18); but no star is needed in the sums over magnetic quantum numbers in eq. (10).

B The matrix $D^{(1)}$(u)

This appendix calculates the rows of $D^{(1)}(u)$, which form a natural basis for the vector operators $\tilde{\mathrm{E}}$ and h. More precisely, from eq. (14), the relevant basis vectors are

$$\exp[\mathrm{i}\,\beta\,\mathrm{B}]\,\mathrm{D}^{(1)}(\mathrm{u})_{\mathrm{BA}};\mathrm{A},\mathrm{B}=+1,0,-1.$$

The subscripts (B,A) give the components A of unit vector B.

One may evaluate the components of each vector, starting from

$$\mathrm{D}^{(1)}(\mathrm{u})_{\mathrm{BA}}=\exp[\mathrm{i}(\beta-\pi/2)(\mathrm{A}-\mathrm{B})]\mathrm{d}^{(1)}(\alpha/2)_{\mathrm{BA}},$$

with

$$\mathrm{d}^{(1)}(\alpha/2)=\begin{bmatrix}(1+\cos)/2 & +\sin/\sqrt{2} & (1-\cos)/2\\ -\sin/\sqrt{2} & \cos & +\sin/\sqrt{2}\\ (1-\cos)/2 & -\sin/\sqrt{2} & (1+\cos)/2\end{bmatrix}; \qquad (67)$$

$\cos=\cos(\alpha/2),\sin=\sin(\alpha/2)$; rows and columns are labeled with spherical components in the order (+1, 0, -1).

From eq. (8), the basis vector B = 0 is just the unit vector $\hat{V}$(u), i. e. $\hat{V}$(h) with h = u. This vector has spherical and Cartesian components

$$\begin{aligned}\hat{V}(u)_A &= D^{(1)}(u)_{0A}\\ &= (\mp\sin(\alpha/2)\exp[\pm i(\beta-\pi/2)]/\sqrt{2},\cos(\alpha/2))\\ &= (\sin(\alpha/2)\sin(\beta),-\sin(\alpha/2)\cos(\beta),\cos(\alpha/2)). \qquad (68)\end{aligned}$$

Spherical components are listed in the order (±,0); Cartesian components in the order (X,Y,Z).

As for the rows $B=\pm1$, it is best to form linear combinations in order to get real vectors. One linear combination turns out to be $\hat{n}$, the axis of rotation for u, while the orthogonal linear combination turns out to be $\hat{n}\times\hat{V}$, the third axis of an orthogonal coordinate system $(\hat{n},\hat{V},\hat{n}\times\hat{V})$.

$$
\begin{aligned}
\sqrt{2}\hat{n}_A &= -\exp[+i\beta]D^{(1)}(u)_{+1,A} + \exp[-i\beta]D^{(1)}(u)_{-1,A} \\
&= -i(d^{(1)}_{+1,A} + d^{(1)}_{-1,A})\exp[iA(\beta-\pi/2)] \\
&= (\mp\exp[\pm i\beta], 0) \\
&= \sqrt{2}(\cos(\beta), \sin(\beta), 0);
\end{aligned} \tag{69}
$$

$$
\begin{aligned}
i\sqrt{2}(\hat{n}\times\hat{V})_A &= -\exp[+i\beta]D^{(1)}(u)_{+1,A} - \exp[-i\beta]D^{(1)}(u)_{-1,A} \\
&= -i(d^{(1)}_{+1,A} - d^{(1)}_{-1,A})\exp[iA(\beta-\pi/2)] \\
&= i(\mp\cos(\alpha/2)\exp[\pm i(\beta-\pi/2)], -\sqrt{2}\sin(\alpha/2)) \\
&= i\sqrt{2}(\cos(\alpha/2)\cos(\beta-\pi/2), \cos(\alpha/2)\sin(\beta-\pi/2), -\sin(\alpha/2)).
\end{aligned} \tag{70}
$$

For convenience I also record the inverses of eqs. (69) and (70).

$$
\begin{aligned}
D^{(1)}(u)_{0A} &= \hat{V}(u)_A; \\
\exp[+i\beta]D^{(1)}(u)_{+1,A} &= (-\hat{n}_A - i(\hat{n}\times\hat{V})_A)/\sqrt{2}; \\
\exp[-i\beta]D^{(1)}(u)_{-1,A} &= (\hat{n}_A - i(\hat{n}\times\hat{V})_A)/\sqrt{2}.
\end{aligned} \tag{71}
$$

The vector $\hat{V}$(u) will turn out to be the peak value of $\hat{V}$(h); therefore (since the relation between $\hat{V}$ and $h^{(1/2)}$ is invertible) $u^{(1/2)}$ will be the peak value of $h^{(1/2)}$. We could have used the notation $\hat{r}$ for the unit vector which we labeled $\hat{V}$(u), because the peak value of radius (for the fictional electron moving in a central force) is given by the vector $\hat{V}$(u).

C The matrix $D^{(L)}$(H)

This appendix derives the properties of the Hermitean factors D(H) occurring in the coherent state. The initial formulas will be valid for general $\hat{p}$ and p; later results will use the assumptions $\hat{p} = (\cos(\beta + \mu), \sin(\beta+\mu), 0)$; i. e. $p_3 = 0$; and $e^p \gg 1$.

Most of the mathematical techniques used in this appendix are a direct steal from Thiemann-Winkler paper II [12], with one significant exception: I make no use of traces. Because Thiemann

and Winkler deal with the general SU(2) expansion (matrices D_{MN}, both M and N summed over) they are able to recast their results for D(H) as theorems about class invariants, the traces D_{MM} . In my case the expansion matrices are D_{0M}, sum over M only; I have not been able to recast my results as theorems about traces. Instead, in order to obtain manageable forms for D(H), I use the assumption $e^p \gg 1$. This assumption is not a serious limitation unless one wishes to extend the calculation to very low values of L of order 10. (For discussion of this point, see the estimates given for the size of t, in appendix D. These estimates in effect also limit the size of p.)

To obtain an approximate form for D(H), I write down the power series expansion for D(H), then use the $e^p \gg 1$ assumption to simplify the series. Start from $D(H)^{(1/2)}$, which has the form

$$\begin{aligned} D(H)^{1/2} &= \exp[\,\vec{p}\cdot\vec{S}\,] \\ &= \begin{bmatrix} \cosh(p/2)+\hat{p}_3\sinh(p/2) & \sinh(p/2)(\hat{p}_1-i\hat{p}_2) \\ \sinh(p/2)(\hat{p}_1+i\hat{p}_2) & \cosh(p/2)-\hat{p}_3\sinh(p/2) \end{bmatrix} \\ &:= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{aligned} \tag{72}$$

(Despite the superscript 1/2, this is a matrix in O(3).) Given the abcd, the $D(H)^{(L)}$ for arbitrary L may be computed from the series

$$\begin{aligned} &D^{(L)}(H)_{MN} = \\ &\quad (ad)^L(a/b)^N(b/d)^M\sqrt{(L+M)!(L-M)!(L+N)!(L-N)!} \\ &\qquad \cdot\,\Sigma_k\frac{(bc/ad)^k}{(L-M-k)!(L+N-k)!(M-N+k)!k!}. \end{aligned} \tag{73}$$

This formula is valid for H an arbitrary complex matrix in SL(2,C), not just for H a Hermitean matrix.

A Large p limit of D(H)

We now assume e^p large; $\hat{p}_3$ may be large, but not too close to ± 1: $(1-\mid\hat{p}_3\mid) \gg 2e^{-p}$.) Then

$$
\begin{aligned}
\cosh(p/2) &\cong \sinh(p/2) \\
&\cong e^{p/2}/2; \\
ad &\cong bc; \\
D^{(L)}(H)_{MN} &\cong (ad)^L (a/b)^N (b/d)^M \sqrt{(L+M)!(L-M)!(L+N)!(L-N)!} \\
&\quad \cdot \Sigma_k [(L-M-k)!(L+N-k)!(M-N+k)!k!]^{-1}. \qquad (74)
\end{aligned}
$$

The series in k can be summed using the addition theorem for binomial coefficients:

$$
\sum_{\mathrm{k}} \left(\begin{array}{c} \mu! \\ k \end{array} \right) \left(\begin{array}{c} \nu! \\ \lambda - k \end{array} \right) = \left(\begin{array}{c} (\mu+\nu)! \\ \lambda! \end{array} \right) \qquad (75)
$$

The eqs. (74) and (75) give

$$
\begin{aligned}
D^{(L)}(H)_{MN} &\cong (ad)^L (a/b)^N (b/d)^M \frac{(2L)!}{\sqrt{(L+M)!(L-M)!(L+N)!(L-N)!}} \\
&= (\hat{p}_1 - i\hat{p}_2)^{M-N} (\exp(p/2)/2)^{2L} [1+\hat{p}_3]^{L+N} [1-\hat{p}_3]^{L-M} \\
&\quad \cdot \frac{(2L)!}{\sqrt{(L+M)!(L-M)!(L+N)!(L-N)!}}. \qquad (76)
\end{aligned}
$$

At this point one can prove: let $\bar{M}$ and $\bar{N}$ denote the peak values of M and N, i.e. the values which maximize $\mid D(H) \mid$. Then

$$
\bar{\mathrm{M}}/\mathrm{L} \cong \bar{\mathrm{N}}/\mathrm{L} \cong \hat{\mathrm{p}}_3. \qquad (77)
$$

Proof: to find the peak value of (say) N, compute the first difference of the square magnitude of the N dependence of D(H), and set this first difference equal to zero.

$$
\begin{aligned}
\delta^{(1)} f(N) &:= f(N+1) - f(N); \\
f(N) &= \mid a/b \mid^{2N} / [(L+N)!(L-N)!]. \\
0 &= \delta^{(1)} f(\bar{N}) \\
&\propto \mid a/b \mid^2 \frac{L - \bar{N}}{L + \bar{N} + 1} - 1; \\
\bar{N}/L &\cong \frac{\mid a \mid^2 - \mid b \mid^2}{\mid a \mid^2 + \mid b \mid^2}; \\
\bar{N}/L &\cong \hat{p}_3. \qquad (78)
\end{aligned}
$$

On the last line I have used the values of abcd from eq. (72). The proof for $\bar{M}$ is identical except for the replacements (a,b) →(b,d). □

B Small p_3 limit of D(H)

In the main body of the text I focus on the case $p_3 = 0$. From eq. (77) of the last subsection, in this limit the important values of M and N satsify $L >> M, N$. Therefore one can use Stirling's approximation for the factorials in eq. (76), for example

$$\frac{(2\mathrm{L})!}{(\mathrm{L}+\mathrm{M})!(\mathrm{L}-\mathrm{M})!} \cong \frac{(2\mathrm{L}^{2\mathrm{L}}}{\sqrt{\pi}(\mathrm{L}-\mathrm{M})^{\mathrm{L}-\mathrm{M}+1/2}(\mathrm{L}+\mathrm{M})^{\mathrm{L}+\mathrm{M}+1/2}}. \quad (79)$$

Now use

$$\begin{aligned} (1+x/n)^n &= \exp[n \ln(1+x/n)] \\ &\cong \exp[x - x^2/2n + \ldots]. \end{aligned} \quad (80)$$

Take $n = L + 1/2$, $x = \pm M$. Also, write L^{2L} as $(L + 1/2 - 1/2)^{2L}$ and apply eq. (80) to this factor.

$$\frac{(2\mathrm{L})!}{(\mathrm{L}+\mathrm{M})!(\mathrm{L}-\mathrm{M})!} \cong \frac{(2^{2\mathrm{L}} \exp[-\mathrm{M}^2/(\mathrm{L}+1/2)]}{\sqrt{\pi(\mathrm{L}+1/2)}} \quad (81)$$

To obtain a result valid near $p_3 = 0$, as well as at $p_3 = 0$, assume $\hat{p}_3 \leq$ order $1/\sqrt{L+1/2}$. Then apply eq. (80) to the $[1 \pm \hat{p}_3]$ factors, with now n = L+1/2, x = $\pm\hat{p}_3(L + 1/2)$. For example,

$$\begin{aligned} [1+\hat{p}_3]^{L+N} &= [1+\hat{p}_3(L+1/2)/(L+1/2)]^{(L+1/2)[1+(N-1/2)/(L+1/2)]} \\ &\cong \exp[\hat{p}_3(L+1/2)+\hat{p}_3(N-1/2) \\ &\quad -(\hat{p}_3)^2(L+1/2)/2], \end{aligned} \quad (82)$$

and similarly for the $[1-\hat{p}_3]$ factor. Inserting eqs. (81) and (82) into eq. (76) yields

$$\begin{aligned} D(H)^{(L)}_{MN} &\cong (\hat{p}_1 - i\hat{p}_2)^{M-N} \frac{\exp(pL)}{\sqrt{\pi(L+1/2)}} \\ &\quad \cdot \exp\{-[M - \hat{p}_3(L+1/2)]^2/2(L+1/2)\} \\ &\quad \cdot \exp\{-[N - \hat{p}_3(L+1/2)]^2/2(L+1/2)\} \end{aligned} \quad (83)$$

The M (and N) dependence of D(H) is peaked at $M = \hat{p}_3(L+1/2)$, with the squared width of the Gaussian equal to $\sqrt{L(L+1)} \cong (L+1/2)$. This is already a bit more than we need for the main body of the paper.

Eq. (83) demonstrates Gaussian behavior in N and M. To obtain Gaussian behavior in L, multiply D(H) by the other exponential factor in the coherent state.

$$\begin{aligned} &\exp(-tL(L+1)/2)D(H)^{(L)}_{NM} \cong \exp(-tL(L+1)/2)e^{Lp}\cdots \\ &\quad = \exp[-t((L+1/2)-p/t)^2/2 + p^2/(2t) + t/8 - p/2]\cdots . \end{aligned} \tag{84}$$

The $\cdots$ indicates irrelevant factors which are bounded for large L. On the last line one can neglect $\exp(t/8) \cong 1$. Eq. (84) is a Gaussian in L with mean $< L+1/2 >= p/t$ and standard deviation $1/\sqrt{t}$.

Since t is small, the standard deviation is very large. However, what counts is $\sigma_L / < L+1/2 >= \sqrt{t}/p$, which is small as required.

When $p_3 = 0$, at eq. (7) I have introduced the notation

$$\hat{\mathrm{p}} = (\cos(\beta+\mu), \sin(\beta+\mu), 0)$$

for $\hat{p}$. For this value of $\hat{p}$, D(H) becomes

$$\begin{aligned} &\exp(-tL(L+1)/2)D(H)^{(L)}_{MN} \cong \exp[-t((L+1/2)-p/t)^2/2] \\ &\cdot \exp[-M^2/2(L+1/2)]\exp[-N^2/2(L+1/2)][1/\sqrt{\pi(L+1/2)}] \\ &\cdot \exp[p^2/(2t) - p/2](\exp[-i(\beta+\mu)])^{M-N}. \end{aligned} \tag{85}$$

D Small Correction (SC) Terms

This appendix discusses the nature of the small corrections SC,

$$\text{operator} \mid \text{coh state}\rangle = \text{<operator>} \mid \text{coh state}\rangle + \text{SC}.$$

The first subsection constructs a set of states which are orthonormal, and uses them to expand the SC for the $\tilde{\mathrm{E}}$ operator. The coefficients multiplying these states are shown to be suppressed by factors involving the small parameters $1/\sqrt{< L >}$ and $\sqrt{t}$. A second section shows that the SC terms emphasize values of the parameter near,

but not at the average value. The term "near coherent" is introduced to describe this behavior. A third subsection introduces a set of states appropriate for expanding the SC terms for the holonomy operator. A final subsection argues that the parameter t should be taken to be order $1/<L>$ in order to minimize the size of the SC terms. If t is replaced by a number of order $1/<L>$, everywhere in the factors multiplying the SC terms, then all the SC terms turn out to be suppressed by factors of the same order, $1/\sqrt{<L>}$.

A SC States for the $\tilde{\mathrm{E}}$ Operator

At eq. (15) I replaced

$$\mathrm{L} \rightarrow <\mathrm{L}> + \Delta\mathrm{L};\ \mathrm{M} \rightarrow \Delta\mathrm{M},$$

then asserted that the terms proportional to $<L>$ represented the dominant contribution. I must now examine the terms involving ΔX, X = L or M, and show that they are small.

From the previous appendix, eq. (85), the original coherent state is proportional to Gaussian factors coming from the D(H) factor. Therefore the SC terms are proportional to the first moments of these Gaussians.

$$\begin{aligned} \mid u, \vec{p}\rangle &\propto \sum_{L,M} D^{(L)}(hu\dagger)_{0M} \\ &\quad \cdot \exp[-t((L+1/2)-p/t)^2/2]\exp[-M^2/2(L+1/2)]; \\ \text{SC terms} &\propto \sum_{L,M} D^{(L)}(hu\dagger)_{0M}[\Delta L \,\mathrm{or}\, \Delta M] \\ &\quad \cdot \exp[-t((L+1/2)-p/t)^2/2]\exp[-M^2/2(L+1/2)]. \end{aligned}$$

It is difficult to carry out the sums over L and M, because the $D^{(L)}(hu\dagger)_{0M}$ factor is difficult to approximate and may have complicated L and M dependence.

There is a simpler way to estimate the order of magnitude of the SC terms, without knowing in detail the M and L dependence of $D^{(L)}(hu\dagger)_{0M}$. For the SC terms involving M and ΔL, define the states

$$\begin{aligned} \mid m1(M)\rangle &:= N(m1(M))\sum_{L,M}((2L+1)/4\pi)\exp[-tL(L+1)/2] \\ &\qquad \cdot [D^{(L)}(hu\dagger)_{0M} M D^{(L)}(H)]_{M0}; \end{aligned} \tag{86}$$

$$
\begin{aligned}
\mid m1(L)\rangle \;\; &:= \;\; N(m1(L))\sum_{L,M}[(2L+1)/4\pi]\exp[-tL(L+1)/2] \\
&\cdot[D^{(L)}(hu\dagger)_{0M}\,\Delta L\, D^{(L)}(H)]_{M0}. \qquad (87)
\end{aligned}
$$

The notation mp(X) denotes the pth moment of the variable X. The above states are identical to the original coherent state $\mid u,\vec{p}\rangle$ except for a different normalization factor,

$$
\mathrm{N}\rightarrow\mathrm{N}(\mathrm{m1}(\mathrm{X})),
$$

and one power of ΔX in the summand. In terms of these states, eq. (15) becomes

$$
\begin{aligned}
(\gamma\kappa/2)^{-1}\tilde{\mathrm{E}}^x_A \mid u,\vec{p}\rangle \;\; &= \;\; < L > \; (\hat{n}_A\cos\mu - \hat{n}\times\hat{V})_A\,\sin\mu) \mid u,\vec{p}\rangle \\
&+(N/N(m1(L))) \mid m1(L)\rangle \\
&+(N/N(m1(M))[\hat{V}_A + i\sin\mu\,\hat{n}_A \\
&+i\cos\mu\,(\hat{n}\times\hat{V})_A] \mid m1(M)\rangle. \qquad (88)
\end{aligned}
$$

Evidently this replaces the problem of evaluating the sums over L,M by the problem of determining the normalization ratios N/N(m1(X)). This may seem like replacing Tweedledum by Tweedledee, except the dangerous factors of $D^{(L)}(hu\dagger)_{0M}$ drop out when calculating norms.

I now prove the following:

$$
\begin{aligned}
1 \cong N^2\exp[p^2/t-p]\sqrt{< L+1/2 >}/2\pi\sqrt{t}; \\
N/N(m1(M)) \cong \sqrt{< L+1/2 > /2} : \\
N/N(m1(L)) \cong \sqrt{1/(2t)}. \qquad (89)
\end{aligned}
$$

I begin with the first line of eq. (89). Orthogonality for the D(h) is

$$
\int_{\Omega(\mathrm{h})} \mathrm{D}^{(\mathrm{L})}_{0\mathrm{M}}(\mathrm{h})\mathrm{D}^{(\mathrm{L}')}_{0\mathrm{M}'}(\mathrm{h})^* = \delta_{\mathrm{L},\mathrm{L}'}\delta_{\mathrm{M},\mathrm{M}'}(2\mathrm{L}+1)/4\pi.
$$

From this and eqs. (1) and (3),

$$
\begin{aligned}
1 &= \langle u, \vec{p} \mid u, \vec{p} \rangle \\
&= N^2 \sum_{L,M} \exp[-tL(L+1)][(2L+1)/4\pi] D^{(L)}_{M0}(g\dagger)^* D^{(L)}_{M0}(g\dagger) \\
&= N^2 \sum_{L,M,N,N'} \exp[-tL(L+1)][(2L+1)/4\pi] \\
&\quad \cdot D^{(L)}_{0N}(H) D^{(L)}_{NM}(u) D^{(L)}_{MN'}(u\dagger) D^{(L)}_{N'0}(H) \\
&= N^2 \sum_{L,M} [(2L+1)/4\pi] \exp[-tL(L+1)] \\
&\quad \cdot D^{(L)}_{0M}(H) D^{(L)}_{M0}(H). \qquad (90)
\end{aligned}
$$

The u and h dependence have disappeared.

The next step is to carry out the sum over M. Compare eqs. (4) and (3): when going from $h^{1/2}$ to $H^{(L)}$ we make the replacements

$$\mathrm{i}\,\hat{\mathrm{m}} \cdot \tilde{\mathrm{S}}\,\theta \rightarrow \tilde{\mathrm{S}} \cdot \tilde{\mathrm{p}}.$$

I.e. replace magnitude and direction as follows.

$$\mathrm{i}\,\theta \rightarrow \mathrm{p}; \phi \rightarrow \beta + \mu\, (\Leftrightarrow \tilde{\mathrm{m}} \rightarrow \tilde{\mathrm{p}}).$$

For the angles, see eqs. (5) and (7). Therefore the Euler decomposition of D(H) follows from the Euler decomposition of D(h), eq. (4).

$$\mathrm{D(h)}(-\phi + \pi/2, \theta, \phi - \pi/2) \rightarrow \mathrm{D(H)}(-\beta - \mu + \pi/2, -\mathrm{ip}, \beta + \mu - \pi/2).$$

Therefore the D's on the last line of eq. (90) equal

$$\mathrm{D^{(L)}}(-\beta - \mu + \pi/2, -2\mathrm{ip}, \beta + \mu - \pi/2)_{00}.$$

Approximate this factor using eq. (85) with $p \rightarrow 2p, t \rightarrow 2t$.

$$1 \cong \mathrm{N}^2 \sum_{\mathrm{L}} [(2\mathrm{L}+1)/4\pi] \frac{\exp[-\mathrm{t}((\mathrm{L}+1/2) - \mathrm{p/t})^2 + \mathrm{p}^2/(\mathrm{t}) - \mathrm{p}]}{\sqrt{\pi(\mathrm{L}+1/2)}}. \qquad (91)$$

Replace the sum over L by an integral:

$$
\begin{aligned}
\sum_L (\Delta L = 1) &= (1/\sqrt{t}) \sum_L \Delta(\sqrt{t}(L+1/2) - p/\sqrt{t} := u) \\
&\cong (1/\sqrt{t}) \int du. \qquad (92)
\end{aligned}
$$

Elsewhere in the integral, replace L + 1/2 by its peak value

$$< \mathrm{L} + 1/2 >= \mathrm{p/t}.$$

Eq. (91) then gives the first line of eq. (89).

The remaining two lines of eq. (89) may be proved using similar approximations, with one exception. The calculation of N(m1(M)) resembles the calculation of N, eq. (91), except for an additional factor of M^2 in the summand, so that the sum over M cannot be carried out immediately. Instead, the sum over M may be replaced by an integral over a variable w, using

$$\begin{aligned} \Sigma_M(\Delta M = 1) &= \sqrt{L+1/2}\,\Sigma(\Delta M/\sqrt{L+1/2} := \Delta w) \\ &\cong \sqrt{L+1/2}\int dw. \end{aligned} \tag{93}$$

□

Eq. (89) implies that the SC terms are suppressed: from eq. (88), the leading term is order $< L >$; therefore the SC terms are down by factors of order

$$\begin{aligned} N/N(m1(M)) < L > &= 1/\sqrt{2 < L >}; \\ N/N(m1(L)) < L > &= 1/(\sqrt{2t} < L >). \end{aligned}$$

At first glance the formula for N/N(m1(L)) looks dangerous, because of the small factor of $\sqrt{t}$ in the denominator. However, $t = p/ < L + 1/2 >$, from eq. (85). Therefore

$$\mathrm{N/N(m1(L))}/ < \mathrm{L} > \cong 1/\sqrt{2\mathrm{p} < \mathrm{L} >}.$$

Since I am taking $e^p \gg 1$, both SC terms are down by at least $1/\sqrt{< L >}$.

Section IV constructs a series expansion for the inverse of the volume operator and determines the values of the parameters where the series diverges. To evaluate the terms in the series I need the following theorem, which describes the action of $\tilde{\mathrm{E}}$ on the first moment states.

$$\begin{aligned} \tilde{\mathrm{E}}^x_A \mid m1(M)\rangle &= < \tilde{\mathrm{E}}^x_A >\mid m1(M)\rangle \\ &\quad +b \mid m2(M)\rangle + c \mid u, \vec{p}\rangle \\ &\quad +d \mid m1(M), m1(L)\rangle, \end{aligned} \tag{94}$$

plus a similar theorem for M $\leftrightarrow$ L, where

$$\begin{aligned} b, d &= \text{order}\,[N(m1(M)/N(b,d)]; \\ c &= \text{order}\,[N(m1(M)/N] < L > . \end{aligned} \tag{95}$$

N(b,d) denotes the norm of the state with coefficient b, d.

In words, eqs. (94) and (95) imply that $\mid m1(M)\rangle$ is an approximate eigenfunction of $\tilde{\mathrm{E}}$, with the *same* eigenvalue as $\mid u, \vec{p}\rangle$. Also. the normalization ratios in b, c, d can be evaluated using the same sum-becomes-integral techniques that establish eq. (89). The quantities b, c, d turn out to be order $\sqrt{< L >}$. Therefore the b, c, d terms are small corrections, since $< \tilde{\mathrm{E}}^x_A >$ is order $< L >$ from eq. (9).

The proof of eqs. (94) and (95) for the most part uses the same techniques as the proof for the action of $\tilde{\mathrm{E}}$ on the original ket, eqs. (88) and (89), and there is no point in giving the proof in detail. However, a qualitative sketch of the steps in the proof may be in order, since the c term in eq. (94) is perhaps unexpected. When $\tilde{\mathrm{E}}$ acts on the original ket, it gives back the original ket, or adds a factor of M. In moment notation, the original ket is $\mid m0(M), m0(L)\rangle$. The presence of a c term implies $\tilde{\mathrm{E}}$ can also remove one power of M.

I sketch the proof of eqs. (94) and (95). It is a repeat of the proof for the action of $\tilde{\mathrm{E}}$ on the original ket, eqs. (88) and (89), except for the following changes. In section II, eq. (10), relabel on the left:

$$\mid \mathrm{u}, \tilde{\mathrm{p}}\rangle \rightarrow \mid \mathrm{m1(M)}\rangle \tag{96}$$

On the right, replace the factor of D(H) in the ket by S_ZD(H) (a concise way to replace $D(H)_{M0}$ by M $D(H)_{M0}$). At the next step in the derivation, eq. (11), the M becomes M - B, due to a relabeling M $\rightarrow$ M - B. Therefore in eq. (14), (relabel as at eq. (96), and) replace the C's.

$$\mathrm{C(L,B)} \rightarrow \mathrm{C(L,B)(M-B)}.$$

The derivation now moves to this appendix, eq. (88). Again, relabel on the left, using eq. (96). In the kets on the right, insert an extra (M - B) into the summand for each ket. The M term (in M - B) changes every ket on the right in eq. (88) to a ket with one higher moment.

$$\begin{aligned} \mid u, \vec{p}\rangle &\rightarrow \mid m1(M)\rangle; \\ \mid m1(M)\rangle &\rightarrow \mid m2(M)\rangle; \\ \mid m1(L)\rangle &\rightarrow \mid m1(M), m1(L)\rangle, \end{aligned}$$

(This change is accompanied by a corresponding relabeling of the norms N(i)). The M term therefore yields the leading term in eq. (94), plus the b and d terms.

Next, consider the effect of the -B term in (M - B). Return to eq. (14) for a moment. We are now inserting factors of C(L,M)(-B) into the summand. The -B is just ∓ 1 or 0, independent of M, while the expansion of C(L,M) starts off with a leading $< L >$ term (plus smaller corrections proportional to ΔX, X = L or M). This leading term no longer contains any factor of M. We have *removed* a factor of M, and returned to the original coherent state.

The factor of $< L >$ (in the expansion of C(L,B)) explains the factor of $< L >$ in the order of magnitude given for the coefficient c, eq. (95). The remaining factor in c, the ratio of norms, is given by eq. (89). Despite the factor of $< L >$, c is only order $\sqrt{< L >}$, which makes it a small correction to the leading term, of order $< L >$. The remaining ratios of norms in eq. (95) may be worked out in the same manner as the ratios for the original coherent state. $\Box$

B Near Coherent States

The states $\mid m1(X)\rangle$ emphasize values of X which are near, but not at the average value $< X >$. To see this, note the new state $\mid m1(M)\rangle$ (for example) differs from the original coherent state by the replacement

$$\begin{aligned} D^{(L)}(H)_{M0} &\rightarrow M D^{(L)}(H)_{M0} \\ &\sim M \exp[-M^2/(2\sigma^2)]. \end{aligned}$$

D(H) is Gaussian in M with a peak at M = 0; see eq. (85). Therefore the above function is an odd function of M with a zero (rather than a maximum) at M = 0, a peak located at $M = +\sigma_M$, and a valley located at $M = -\sigma_M$. (From eq. (85), $\sigma_M = \sqrt{L + 1/2}$.)

The shape of this curve suggests that the new coherent state is the difference of two simpler coherent states. In fact,

$$M \exp[-M^2/(2\sigma_M^2)] \;\cong\; (\sigma_M\sqrt{e}/2)\,\{\exp[-(M-\sigma_M)^2/(2\sigma_M^2)] - \exp[-(M+\sigma_M)^2/(2\sigma_M^2)]\,\}. \qquad (97)$$

(To prove eq. (97), multiply out the exponents on the left, and extract common factors. The expression will contain an overall factor of $\sinh(M/\sigma_M)$. Expand this, neglecting terms of order

$$(\mathrm{M}/\sigma_\mathrm{M})^3 = (\mathrm{M}/\sqrt{\mathrm{L}+1/2})^3.$$

These terms are small, because the Gaussians suppress terms with $(M/\sigma_M) \geq 1$. $\Box$

The state $\mid m1(M)\rangle$ has an M distribution which is the *difference* between two very similar Gaussian distributions. This explains why its contribution is so small.

In the main body of this paper I focus on the case $p_3 = 0$. However, in appendix C I discuss briefly coherent states which have small p_3, $L\hat{p}_3 \leq \text{order}\sqrt{L}$. When p_3 is non-zero, the Gaussian in M is peaked at $< M >= \hat{p}_3(L{+}1/2)$ rather than $< M >= 0$. Therefore the Gaussians in eq. (97) correspond to small, non-zero values of $\hat{p}_3$: $\hat{p}_3 = \pm 1/\sqrt{L+1/2}$. The M distribution of $\mid m1(M)\rangle$ emphasizes *small fluctuations* in p_3, near but not at the original mean value $p_3 = 0$.

Similar remarks could be made about the state $\mid m1(L)\rangle$. It is the difference between two coherent states having peak $< L >$ shifted from $< L >$ to $< L > \pm\sigma_L =< L > \pm 1/\sqrt{t}$. Since $< L + 1/2 >= p/t$, these states emphasize small fluctuations in the magnitude p, rather than fluctuations in a component, p_3.

I will refer to states such as $\mid m1(M)\rangle$ and $\mid m1(L)\rangle$ as "near-coherent" states. Whereas the distributions of the coherent state establish a peak value for each variables, the distributions of near-coherent states emphasize the fluctuations near the peak value.

In a certain sense the near-coherent states are coherent. If a holonomy or $\tilde{\mathrm{E}}$ operator is applied to the near-coherent state, the near coherent state is found to be an approximate eigenvector with the *same* approximate eigenvalue as the coherent state. Even though the distribution for the near-coherent state may have a zero where the coherent state has a peak, the eigenvalue ($\cong$ average value) is unchanged, because the off-center Gaussian peaks of the near-coherent

state are symmetrically distributed around the original peak value. This is the qualitative explanation for the $< \tilde{\mathrm{E}}^x_A >$ term in eq. (94).

C SC States for the Holonomy

As with the $\tilde{\mathrm{E}}$ operators, I estimate the order of magnitude of the SC terms by calculating appropriate norms. Since the holonomy produces states containing $D^{(L')}(h)$ with $L' = L \pm 1$, the states $\mid m1(X)\rangle$ are not enough, and I will need the following additional states:

$$\begin{aligned}
\mid L_\pm\rangle &:= N(L_\pm)\sum_{L,M}((2L+1)/4\pi)\exp[-tL(L+1)/2] \\
&\quad \cdot(1/2)[D^{(L+1)}(hu\dagger) \pm D^{(L-1)}(hu\dagger)]_{0M}D^{(L)}(H)_{M0}; \\
\mid L+1, m1(M)\rangle &:= N(L+1, m1(M))\sum_{L,M}((2L+1)/4\pi)\exp[-tL(L+1)/2] \\
&\quad \cdot D^{(L+1)}(hu\dagger)_{0M}MD^{(L)}(H)_{M0}. \qquad (98)
\end{aligned}$$

In terms of these states, eq. (28) becomes

$$\begin{aligned}
D^{(1)}(h)_{0,A} \mid u, \vec{p}\rangle &\cong N/N(L_+) \mid L_+\rangle\hat{V}_A \\
&\quad +N/N(L+1, m1(M)) \mid L+1, m1(M)\rangle \\
&\quad \cdot[-\cos\mu\,\hat{n}_A + \sin\mu\,(\hat{n}\times\hat{V})_A]/ < 2L+1) > \\
&\quad -iN/N(L_-) \mid L_-\rangle[\sin\mu\,\hat{n}_A + \cos\mu\,(\hat{n}\times\hat{V})_A] \\
&\quad +\text{order}\,(M/L)^2, 1/L). \qquad (99)
\end{aligned}$$

The order of magnitude of the SC terms may be determined by computing the following norms:

$$\begin{aligned}
N/N(L_+) &\cong 1; \\
N/N(L_-) &\cong \sqrt{t/2}; \\
N/N(L+1, m1(M)) &\cong \sqrt{< L+1/2 > /2}. \qquad (100)
\end{aligned}$$

Proof: the last line of eq. (100) is easiest to establish. Because $\mid$ L+1,m1(M)$\rangle$ differs from $\mid m1(M)\rangle$ only in the replacement of $D^{(L+1)}(hu\dagger)$ by $D^{(L)}(hu\dagger)$, and the $D(hu\dagger)$ factors drop out anyway

when computing norms, N/N(L+1,m1(M)) is the same as N/N(m1(M)), eq. (89).

To determine the N($L_\pm$), rewrite the $L_\pm$ states as follows.

$$\begin{aligned}
\mid L_\pm\rangle &= N(L_\pm)\sum_{L,M}((2L+1)/4\pi)\exp[-tL(L+1)/2] \\
&\quad \cdot(1/2)[D^{(L+1)}(hu\dagger) \pm D^{(L-1)}(hu\dagger)]_{0M}D^{(L)}(H)_{M0} \\
&\cong N(L_\pm)\sum_{L,M}((2L+1)/4\pi)D^{(L)}(hu\dagger)_{0M} \\
&\quad \cdot(1/2)\sum_{\pm}[(\pm 1)\exp\{-t[L\mp 1+1/2-p/t]^2/2\}] \\
&\quad \cdot\exp\{p^2/2t-p/2-iM(\beta+\mu)\}\frac{\exp\{-M^2/[2(L+1/2)]\}}{\sqrt{\pi(L+1/2)}} \\
&\cong N(L_\pm)\sum_{L,M}((2L+1)/4\pi)D^{(L)}(hu\dagger)_{0M} \\
&\quad \cdot\exp\{-t[L+1/2-p/t]^2/2\}\left[\begin{array}{c}\cosh[t(L+1/2-p/t)] \\ \sinh[t(L+1/2-p/t)]\end{array}\right] \\
&\quad \cdot\exp\{p^2/2t-p/2-iM(\beta+\mu)\}\frac{\exp\{-M^2/[2(L+1/2)]\}}{\sqrt{\pi(L+1/2)}} \\
&\cong N(L_\pm)\sum_{L,M}((2L+1)/4\pi)D^{(L)}(hu\dagger)_{0M} \\
&\quad \cdot\exp[-tL(L+1)/2]D^{(L)}(H)_{M0} \\
&\quad \cdot\left[\begin{array}{c}\cosh[t(L+1/2-p/t)] \\ \sinh[t(L+1/2-p/t)]\end{array}\right]. \qquad (101)
\end{aligned}$$

On the third line I have relabeled $L\pm 1=\tilde{L}$, used eq. (85) to replace the D(H) by Gaussians, and then dropped the tildes. From the last line, the $\mid L_\pm\rangle$ states are just the original states times an additional factor of cosh or sinh. When the state is squared to determine a norm, this factor becomes $\cosh^2[\sqrt{t}u]$ or $\sinh^2[\sqrt{t}u]$ (As at eq. (92), the variable L is replaced by a variable u, and the sum over L is replaced by an integral over u .) For the L_+ state, $\cosh^2[\sqrt{t}u]\cong 1$. The normalization integral for N(L_+) reduces to the normalization integral for N, and we get the first line of eq. (100). For the L$_-$ state, $\sinh^2[\sqrt{t}u]\cong tu^2$. This normalization integral should be compared to the normalization integral for

$| m1(L)\rangle$. That integral contains a $(\Delta L)^2 = u^2/t$ factor. Therefore in $(N/N(m1(L)))^2 = 1/(2t)$, eq. (89), move the t from denominator to numerator to get $(N/N(L_-))^2 = t/2$. □

The states $| L_\pm\rangle$ introduced at eq. (98) are superpositions of the states $| L \pm 1\rangle$ studied in section III.

$$| \mathrm{L}_\pm\rangle = (| \mathrm{L}+1\rangle \pm | \mathrm{L}-1\rangle)/2.$$

In section III it was shown that the states $| L \pm 1\rangle$ are identical to the original coherent state, except that peak angular momentum is shifted by one unit, from $< L >$ (= p/t-1/2) to $< L > \pm 1$. This shift by one unit is very small compared to the width of the Gaussian peak ($1 \ll \sigma_L = 1/\sqrt{t}$). Therefore the superposition which adds the two Gaussians, $| L_+\rangle$, closely resembles the original coherent state. The superposition involving the difference, $| L_-\rangle$, on the other hand, emphasizes fluctuations near the peak in L, rather than the peak. This state is near-coherent, very similar to the state $| m1(L)\rangle$. Consistent with this classification into coherent vs. near-coherent, the normalizations for $| L_\pm\rangle$, eq. (100), make $| L_+\rangle$ dominant and $| L_-\rangle$ a SC.

D Estimates of the parameter t

It is difficult to put significant limits on the parameter t, if one looks only at leading terms. The coefficients of the leading terms are the peak values, and the only peak value (of holonomy, $\tilde{\mathrm{E}}$, L, or angles) which depends on t is $< L >$. From eq. (85) even this peak value depends on t only via p/t, rather than p alone.

$$< \mathrm{L} + 1/2 >= \mathrm{p/t}. \tag{102}$$

To put limits on t, one must consider the SC terms. I list various states contributing to the SC terms. First, the $\tilde{\mathrm{E}}$ SC terms, from eqs. (88) and (89):

$$\sqrt{1/(2 < \mathrm{L} >)}\ | \mathrm{m1(M)}\rangle; \sqrt{1/(2\mathrm{t} < \mathrm{L} >^2)}\ | \mathrm{m1(L)}\rangle.$$

Each state is smaller than the leading term by the factor multiplying the state. Next, the holonomy SC terms, from eqs. (99) and (100):

$$\sqrt{1/4 < \mathrm{L} >}\ | \mathrm{L}+1, \mathrm{m1(M)}\rangle; \sqrt{\mathrm{t}/2}\ | \mathrm{L}_-\rangle$$

One of these terms has t in the numerator, and one has t in the denominator. we can determine a best value of p and t by summing these two factors

$$\sqrt{1/(2\mathrm{t} < \mathrm{L} >^2)} + \sqrt{\mathrm{t}/2},$$

then minimizing the sum with respect to t. The resulting best value is

$$\mathrm{t} = 1/ < \mathrm{L} > . \tag{103}$$

I cannot set $\mathrm{t} = 1/< L >$ exactly, however, because then from eq. (102) I must take p = 1. Appendix C requires an expansion in $\exp(-p) \ll 1$ to simplify D(H). As a compromise, I take p large, but not too large; say p = 5. Then the expansion of D(H) remains valid, since $\exp(-5)$ is small; also the SC terms will be small, provided $< L + 1/2 >$ is large enough. The t dependent factors suppressing the SC terms become

$$\sqrt{1/(2\mathrm{t} < \mathrm{L} >^2)} = \sqrt{1/\mathrm{p} < \mathrm{L} >};\ \sqrt{\mathrm{t}/2} = \sqrt{\mathrm{p}/2 < \mathrm{L} >}.$$

For $< L >$ greater than 100 or so, and p = 5, these factors are sufficiently small.

When the SC terms are taken into account, t is not arbitrarily adjustable. Values of p and t much different from p = 1 and $\mathrm{t} = 1/< L >$ result in larger-than-optimal SC terms.

E Fluctuations in Angles

Given that the matrix h is strongly peaked at the value u it should be possible to translate this into a statement that the angles (θ, ϕ) are strongly peaked at (α, β), then obtain an estimate for the standard deviations of these angles.

To begin, establish the two expectation values

$$0 = \langle u, \vec{p} \mid (D^{(1)}(h) - D^{(1)}(u))_{0A} \mid u, \vec{p}\rangle; \tag{104}$$

$$\begin{aligned} \text{order}\, t, 1/ < L >= \langle u, \vec{p} \mid (D^{(1)}(h)\dagger - D^{(1)}(u)\dagger)_{B0} \\ \cdot(D^{(1)}(h) - D^{(1)}(u))_{0A} \mid u, \vec{p}\rangle. \end{aligned} \tag{105}$$

The first equation is related to a mean; the second to a standard deviation. The first equation is zero because $(D(h) - D(u)) \mid u, \vec{p}\rangle$

contains only SC states, and the leading SC states are odd in one of the variables, M or $L- < L >$, while $\langle u, \vec{p} \mid$ is even.

What happens when a second factor of D(h) - D(u) is applied to the state, as in eq. (105)? The right hand factor $(D^{(1)}(h) - D^{(1)}(u))_{0A} \mid u, \vec{p}\rangle$ is the sum of SC states given at eq. (99),

$$\begin{aligned}
\mid SC_A\rangle &:= (D^{(1)}(h) - D^{(1)}(u))_{0A} \mid u, \vec{p}\rangle \\
&= N/N(L+1, m1(M)) \mid L+1, m1(M)\rangle \\
&\quad \cdot \frac{[-\cos\mu\, \hat{n}_A + \sin\mu\, (\hat{n} \times \hat{V})_A]}{(2L+1)} \\
&\quad -iN/N(L_-) \mid L_-\rangle [\sin\mu\, \hat{n}_A + \cos\mu\, (\hat{n} \times \hat{V})_A]. \qquad (106)
\end{aligned}$$

The left hand factor $\langle u, \vec{p} \mid (D^{(1)}(h)\dagger - D^{(1)}(u)\dagger)_{B0}$ is the Hermitean conjugate $\langle SC_B \mid$. The dot product of the two factors gives vector components of order unity times factors of order

$$\{\mathrm{N}/[\mathrm{N}(\mathrm{L}+1, \mathrm{m1(M)})(2\mathrm{L}+1)]\}^2,\ [\mathrm{N}/\mathrm{N}(\mathrm{L}_-)]^2$$

which are order 1/L and t, respectively, from eq. (100).□

We now have a sharpened version of eq. (105):

$$\begin{aligned}
\langle SC_B \mid SC_A\rangle = \langle u, \vec{p} \mid (D^{(1)}(h)\dagger - D^{(1)}(u)\dagger_{B0}) \\
\cdot (D^{(1)}(h) - D^{(1)}(u))_{0A} \mid u, \vec{p}\rangle, \qquad (107)
\end{aligned}$$

with the left hand side given by eq. (106).

The right hand side is a measure of the standard deviation of the holonomy. To turn this into a measure of angular standard deviation, expand D(h) around D(u). Define

$$\delta\theta = \theta - \alpha;\ \delta\phi = \phi - \beta.$$

Then

$$\begin{aligned}
D(h)_{0A} &= d(\alpha/2 + \delta\theta/2)_{0A} \exp[iS_Z(\beta + \delta\phi - \pi/2)] \\
&= d(\alpha/2)_{0N} \exp[iS_Z(\beta - \pi/2)] \exp[i(\delta\theta/2)\vec{S} \cdot \hat{n}]_{NA} \exp[iS_Z(\delta\phi)] \\
&\cong D(u)_{ON}[1 + i(\delta\theta/2)\vec{S} \cdot \hat{n} + iS_Z(\delta\phi)]_{NA}. \qquad (108)
\end{aligned}$$

The second line uses the behavior of the y axis under rotation around Z,

$$\exp[-\mathrm{iS_Z}(\beta-\pi/2]\,\mathrm{S_Y}\exp[\mathrm{iS_Z}(\beta-\pi/2)] = \tilde{\mathrm{S}}\cdot\hat{\mathrm{n}}.$$

Now insert eq. (108) into eq. (107):

$$\begin{aligned}\langle SC_B \mid SC_A\rangle \cong \langle u,\vec{p}\mid \{D(u)[(\delta\theta/2)\vec{S}\cdot\hat{n} + S_Z(\delta\phi)]\}_{0A}\\ \cdot\{[(\delta\theta/2)\vec{S}\cdot\hat{n} + S_Z(\delta\phi)]D(u)\dagger\}_{B0}\mid u,\vec{p}\rangle.\end{aligned} \tag{109}$$

All matrices D(u) have L = 1. On the right, the $(\delta\theta\,\delta\phi)$ cross terms average to zero, leaving a positive definite expression linear in $(<\delta\theta)^2>, <(\delta\phi)^2>$.

To separate the two standard deviations, first take A = B and sum over A. The matrix D(u) has its axis of rotation along $\hat{n}$, therefore

$$\mathrm{D(u)}\,\tilde{\mathrm{S}}\cdot\hat{\mathrm{n}}\,\mathrm{D(u)}\dagger = \tilde{\mathrm{S}}\cdot\hat{\mathrm{n}}.$$

D(u) rotates the Z axis into the vector $\hat{V}$, therefore

$$\mathrm{D(u)}\,\mathrm{S_Z}\,\mathrm{D(u)}\dagger = \tilde{\mathrm{S}}\cdot\hat{\mathrm{V}}.$$

We get

$$\begin{aligned}\text{l.h.s.} &= \{N/[N(L+1,m1(M))(2L+1)]\}^2 + [N/N(L_-)]^2;\\ \text{r.h.s} &= [<(\delta\theta/2)^2> + \sin^2(\alpha/2) < (\delta\phi)^2>].\end{aligned} \tag{110}$$

The left-hand side of eq. (109) comes from eq. (106). The right-hand side uses

$$\begin{aligned}[(\vec{S}\cdot\hat{V})^2]_{00} &= [L(L+1)/2](V_X^2+V_Y^2)\\ &= [L(L+1)/2]\sin^2(\alpha/2),\end{aligned}$$

plus L = 1; and a similar formula for $(\vec{S}\cdot\hat{n})^2$, except $n_X^2+n_Y^2=1$.

To separate $(<\delta\theta)^2>$ from $<(\delta\phi)^2>$), take A = B = 0 in eq. (109). This kills the $\delta\phi$ terms. In eq. (106) the $\hat{n}$ terms disappear, since this vector has no Z component. The new left- and right-hand sides are

$$\begin{aligned}\text{l.h.s.} &= \sin^2(\alpha/2)\{N/[N(L+1,m1(M))(2L+1)]\}^2\sin^2\mu\\ &\quad + \cos^2\mu\,[N/N(L_-)]^2;\\ \text{r.h.s} &= [<(\delta\theta/2)^2>\sin^2(\alpha/2)].\end{aligned} \tag{111}$$

Eqs. (110) and (111) may be solved for the individual standard deviations.

$$
\begin{aligned}
< (\delta\theta/2)^2 > &= \sin^2\mu(1/(8 < L + 1/2 >) \\
&\quad + \cos^2\mu\,(t/2)]; \\
\sin^2(\alpha/2) < (\delta\phi)^2 > &= \cos^2\mu(1/(8 < L + 1/2 >) \\
&\quad + \sin^2\mu\,(t/2),
\end{aligned}
\tag{112}
$$

where eq. (100) was used to eliminate the ratios of normalizations.

Eq. (112) has the following geometrical interpretation. Sketch the following vectors: draw a vector $\hat{V}$, in a direction $(\theta/2, \phi - \pi/2)$, very near the peak values $(\alpha/2, \beta - \pi/2)$. Draw $\hat{n}$, the axis of u, perpendicular to peak $\hat{V}$ and lying in the XY plane. Draw the third member of the orthogonal triad, $\hat{n} \times \hat{V}$. Draw the angular momentum $< \vec{L} >$. From the electron analogy, this is somewhere in the $\hat{n}$, $\hat{n} \times \hat{V}$ plane, since it must be perpendicular to $\hat{V}$. The exact location of $< \vec{L} >$ is tuned by adjusting the angle μ contained in the vector $\vec{p}$:

$$
< \tilde{\mathrm{E}} > \sim < \tilde{\mathrm{L}} > = < \mathrm{L} > (\hat{\mathrm{n}}_{\mathrm{A}} \cos\mu - \hat{\mathrm{n}} \times \hat{\mathrm{V}})_{\mathrm{A}} \sin\mu),
$$

from eq. (9).

Now suppose, for example, $\sin\mu = 0$, so that $< \vec{L} >$ is along $\hat{n}$. The (t/2) term in eq. (112) comes from fluctuations in magnitude L with the direction $\vec{L}$ fixed, while the $1/ < L + 1/2 >$ term comes from fluctuations in L_Z (fluctuations in direction) with magnitude L fixed. (Cf. eq. (85): the standard deviation for fluctuations in L is $\sqrt{1/t)}$, while the standard deviation for fluctuations in M is $\sqrt{L + 1/2}$.) Suppose first only L fluctuates, with the direction of $< \vec{L} >$ remaining fixed along $\hat{n}$. Then from the sketch, as $L \rightarrow L \pm \Delta L$, the vector $\hat{V}$ will experience rotation along the direction θ: $\theta \rightarrow \theta \pm \delta\theta$. This checks with eq. (112): for $\sin\mu = 0$, a t/2 fluctuation (a fluctuation in the magnitude of L only) causes a fluctuation in θ only.

Similarly, a $1/ < L + 1/2 >$ fluctuation (a fluctuation in direction L_Z, but not in magnitude L) causes a small fluctuation of $\vec{L}$ in a vertical direction. Since $\vec{L}$ must remain perpendicular to $\hat{V}$, the latter fluctuates in an azimuthal direction. Eq. (112) predicts an

azimuthal fluctuation, consistent with this picture. Other values of μ may be handled similarly.

F SC States and Standard Deviations

The uncertainty principle does not allow the SC terms to vanish. Consider, for example, the SC terms generated by the $\tilde{\mathrm{E}}$ operator. If they vanish, then the coherent state is an exact eigenstate of the spin. (The $\tilde{\mathrm{E}}$ operator brings down a factor of spin, therefore is essentially the spin operator.) If spin is exact, then from the uncertainty principle the canonically conjugate variables, the angles θ, ϕ), must be completely uncertain. This implies a completely uncertain holonomy, whereas a coherent state must have both spin and holonomy peaked. Therefore the SC terms cannot vanish.

One can make a more quantitative statement about the SC terms and the uncertainty principle: the SC terms yield the standard deviations of the operators $\tilde{\mathrm{E}}$ and $h^{(1/2)}$ around their means. Proof: let O be an operator which is an approximate eigenvalue of the coherent state,

$$\begin{aligned} O_A \mid u, \vec{p}\rangle &= < O_A > \mid u, \vec{p}\rangle + SC \\ &= < O_A > \mid u, \vec{p}\rangle + \sum_i \hat{e}_{iA} \lambda_i \mid nc_i\rangle. \end{aligned} \tag{113}$$

I have given O a vector subscript because the important operators in this paper are vectors; but this feature is unimportant. The $\hat{e}_i$ are unit vectors. When the standard deviation is computed, the original coherent state drops out, and only the near coherent, SC terms contribute:

$$\begin{aligned} \langle u, \vec{p} \mid (O_A - < O_A >)^2 \mid u, \vec{p}\rangle &= \sum_{ij} \hat{e}_{jA}^* \hat{e}_{iA} \lambda_j^* \lambda_i \langle nc_j \mid nc_i\rangle \\ &= \sum_i \hat{e}_{iA}^* \hat{e}_{iA} \lambda_i^* \lambda_i, \end{aligned} \tag{114}$$

On the last line I have used the orthogonality of the near coherent states. From eq. (114), only the near coherent states contribute to the total standard deviation squared.

From the discussion of SC states in appendix D, e. g. eq. (88), each λ_i will contain a factor of N/N(i). The above discussion clarifies why each N/N(i) contains a factor which may be interpreted as a standard deviation, for either the L or M distribution. Eq. (114) expresses the total squared standard deviation as the sum of individual squared standard deviations $\mid \lambda_i \mid^2$ contributed by each near coherent state. In statistics, the sum-of-squares form is obtained when the fluctuations in a total quantity are caused by fluctuations in several variables which are statistically independent.

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